Contents

Class 1 – Review

Set: is a collection of objects, called elements, enclosed in braces.

The following are examples of sets

- **Natural numbers**: $\{1, 2, 3, 4, ...\}$
- **Whole numbers**: $\{0, 1, 2, 3, 4, ...\}$
- **Integers**: $\{..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$
- **Rational numbers** are real numbers that can be written as the quotient of two integers: ex: $\frac{2}{1}, \frac{3}{7}, \frac{13}{4}, etc$
- **Irrational numbers** are real numbers that cannot be expressed as the quotient of two integers $\pi, \sqrt{2}, \sqrt{7}, etc$
- **Real numbers** are (all of the above) all numbers that correspond to a point in the real number line.

  Practice: *Classify the following numbers*

  1. 4
  2. 3.2
  3. $\sqrt{3}$
  4. $\sqrt{9}$

- **Prime number** is a number that is divisible only by 1 and itself.

  Practice: Give examples of prime numbers ______________________________

- **Absolute Value** of a number a, denoted by $|a|$, is the distance of a from the 0 in the number line.

  Example:

  5. $|4| =$
  6. $|-3| =$

Fractions

A **fraction** is a quotient of two integers $\boxed{\frac{a}{b}}$ ←Numerator ←Denominator $Ex: \frac{2}{3}, \frac{7}{4}, \frac{1}{12}$ *etc.*,

- ***Simplifying:*** write a fraction in lowest terms (i.e., the Num. & Denom. have no common factors)

  To simplify a fraction we factor the Num. and Denom. and cancel like terms

  Example: $\frac{15}{12} = \frac{\cancel{3}\cdot 5}{\cancel{3}\cdot 4} = \frac{5}{4}$

  Here $\frac{15}{12}$ and $\frac{5}{4}$ are equivalent fractions since they are the same quantity

- ***Reciprocal*** of a fraction is another fraction s.t. when their product is equal to 1.

  Example: *The reciprocal of* $\frac{5}{2}$ *is* $\frac{2}{5}$ *since* $\frac{5}{2}\cdot\frac{2}{5} = 1$

  So, to find the reciprocal of a fraction we flip it

  - ***Multiply:*** $\boxed{\frac{a}{b}\cdot\frac{c}{d} = \frac{a\cdot c}{b\cdot d}}$ *Example:* $\frac{11}{2}\cdot\frac{4}{3} = \frac{11\cdot \cancel{4}^{2}}{\cancel{2}\cdot 3} = \frac{11\cdot 2}{3} = \frac{22}{3}$

  So, to multiply two fractions, we muliply the Num. together and the Denom. together..

  - ***Divide:*** $\boxed{\frac{a}{b}\div\frac{c}{d} = \frac{a}{b}\cdot\frac{d}{c} = \frac{ad}{bc}}$ *Example:* $\frac{11}{3}\div\frac{5}{6} = \frac{11}{3}\cdot\frac{6}{5} = \frac{11\cdot \cancel{6}^{2}}{5\cdot \cancel{3}} = \frac{11\cdot 2}{5} = \frac{22}{5}$

  To divide two fractions, we flip the second fraction and then muliply the two fractions

- ***Add/Subtract:*** $\boxed{\frac{a}{b}\pm\frac{c}{d} = \frac{a}{b}\cdot\frac{d}{d}\pm\frac{c}{d}\cdot\frac{b}{b} = \frac{ad\pm cb}{bd}}$

  Example: $\frac{11}{3}\pm\frac{5}{2} = \frac{11}{3}\cdot\frac{2}{2}\pm\frac{5}{2}\cdot\frac{3}{3} = \frac{22}{6}\pm\frac{15}{6} = \frac{22\pm 15}{6} = \frac{37}{6}$

To add or subtract fractions they have to have the same denominator.

Practice: *Perform the following operations.*

7. $\dfrac{5}{11}+\dfrac{7}{11}=$

8. $\dfrac{3}{4}-\dfrac{5}{7}=$

9. $\dfrac{5}{12}+\dfrac{3}{20}=$

10. $\dfrac{12}{28}+\dfrac{5}{24}=$

Exponential Expression

An exponential expression is an expression of the form $\underset{\leftarrow\,base}{a}^{n\leftarrow Exponent} = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n-times}$

Ex: $4^3 = 4 \cdot 4 \cdot 4 = 64$, (-2)^4 = (-2)(-2)(-2)(-2) = 16

Practice: *Simplify*

11. $7^1 =$

12. $\left(\frac{2}{3}\right)^3 =$

13. $(-3)^4 =$

14. $-5^2 =$

Exponential Rules:

- $a^x \cdot a^y = a^{x+y}$ $\quad (ab)^x = a^x b^x$ $\quad (a+b)^x \neq a^x + b^x$
- $\frac{a^x}{a^y} = a^{x-y}$ $\quad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$ $\quad a^0 = 1$
- $(a^x)^y = a^{xy}$ $\quad a^{-x} = \frac{1}{a^x}$ $\quad a^{x/y} = \sqrt[y]{a^x}$

Practice:

1. $\left(\frac{r}{s}\right)^6 =$

2. $\left(\frac{5x^6}{9y^3}\right)^2 =$

3. $\left(\frac{9x^3}{y}\right)^{-2} =$

4. $\frac{6^{-5}x^{-1}y^2}{6^{-2}x^{-4}y^4} =$

5. $\frac{(3x^{-2}y)^{-2}}{4x^7y} =$

Order of Operation

P-E-M-D-A-S

- Start with grouping symbols, like parenthesis or absolute values
- Simplify exponential expressions
- Multiply or Divide whichever comes first from left to right
- Add or subtract whichever comes first from left to right

Example: *Evaluate the following.*

15. $\dfrac{8\cdot 2^3 - |3-7| + 5(3^2 - 4)}{5\cdot 2 - 7} =$

- **Scientific Notation:**

A number is written in scientific notation if it is written as the product of a number α, where $1 \le \alpha < 10$ and an integer power r of 10. $\boxed{\alpha \times 10^r}$

Example: 1.3×10^5

16.Practice: *Write the number in Scientific Notation*

9,060,000,000 =

0.00017 =

17. Practice: *Write the number in Decimal Notation*

$3.067 \times 10^{-4} =$

$6.002 \times 10^{6} =$

18. Practice: *Perform each operation and write in standard form and Scientific Notation*

$(9 \times 10^7)(4 \times 10^{-9})$

$\dfrac{8\times 10^4}{2\times 10^{-3}} =$

- **Products/Quotients involving Zeroes:**

- $\boxed{b \cdot 0 = 0}$
- $\boxed{\frac{N}{0} \rightarrow undefined}$
- $\boxed{\frac{0}{K} = 0}$

- **Properties of Real Numbers**

	Additive	Multiplicative
• **Commutative:**	$a+b=b+a$	$a \cdot b = b \cdot a$
Example:	$3+2=2+3$	$3 \cdot 2 = 2 \cdot 3$

i.e., if we change the order of numbers, the result doesn't change

• **Associative:**	$(a+b)+c=a+(b+c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Example:	$(3+2)+5=3+(2+5)$	$(3 \cdot 2) \cdot 5 = 3 \cdot (2 \cdot 5)$

i.e., if we change the grouping of numbers, the result doesn't change

- **Distributive:** $a(b+c)=ab+ac$

Example: $5(3+2)=5 \cdot 3+5 \cdot 2=15+10=25$

Practice: Use the commutative property to rewrite the following expressions.

19. $(6+2)+3=$

20. $5 \cdot (2 \cdot 7)=$

Practice: Use the associative property to rewrite the following expressions.

21. $8+(3+4)=$

22. $a(bc)=$

23. $(9 \cdot 2) \cdot 5=$

Radicals:

In General: ***b* is the square root of a number *a* if $b^2 = a$.**

Notation: $\sqrt{a} = b$, $\sqrt{\ }$ - radical sign, a - radicand

Examples: 1. $\sqrt{100} =$ 2. $-\sqrt{36} =$ 3. $\sqrt{1} =$ 4. $\sqrt{\frac{25}{81}} =$

Note: A square root of a negative number is **NOT** a real number

For any real number *a*,
$\sqrt{a^2} = |a|$

$Ex: \sqrt{x^2} = |x|, \quad \sqrt{(-8)^2} = |-8|, \quad \sqrt{(7y)^2} = |7y|$

Product Rule: *If* $\sqrt[n]{a}$ *and* $\sqrt[n]{b}$ *are real numbers, then* $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

Quotient Rule: *If* $\sqrt[n]{a}$ *and* $\sqrt[n]{b}$ *are real numbers and* $b \neq 0$, *then* $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

- **Simplify:** In order to simplify you are looking to rewrite the radicand as a product of a perfect power times something else.

  Examples: $\sqrt{4x^6} = \sqrt{4}\sqrt{\left(x^3\right)^2} = 2x^3$, $\quad \sqrt{\frac{16y^{24}}{81}} = \frac{\sqrt{16y^2}}{\sqrt{81}} = \frac{4y}{9}$

- **Add/Subtract: Combine like radicals** (radical expressions that have the same index and the same radicand).

  $* \quad \sqrt{28} + \sqrt{63} = \sqrt{4 \cdot 7} + \sqrt{9 \cdot 7} = 2\sqrt{7} + 3\sqrt{7} = 5\sqrt{7}$

  Examples

  $* \quad \sqrt{72x^2} + \sqrt[3]{54} - x\sqrt{50} - 3\sqrt[3]{2} = \sqrt{36x^2 \cdot 2} + \sqrt[3]{27 \cdot 2} - x\sqrt{25 \cdot 2} - 3\sqrt[3]{2} =$

  $6x\sqrt{2} + \cancel{3\sqrt[3]{2}} - 5x\sqrt{2} - \cancel{3\sqrt[3]{2}} = 6x\sqrt{2} - 5x\sqrt{x} = x\sqrt{2}$

- **Multiply.** In order to multiply radicals the roots have to have the same index, then multiply coefficients together and radicands together.

  Examples $* \quad 2\sqrt{3} \cdot 7\sqrt{5} = \sqrt{3 \cdot 5} = 14\sqrt{15}$, $\quad \sqrt[4]{8} \cdot \sqrt[4]{2} = \sqrt[4]{16} = 2$

More Definitions:

- **Variables** are symbols used to represent an unknown quantity.

Example: $x,\ y,\ t,$ can you give any more examples?

- **Algebraic Expression** is a collection of numbers, variables and operating and grouping symbols.

Example: $5x,\quad 3(y-2),\quad \dfrac{y^2-2y+1}{3},$ can you give any more examples?

*We can **Evaluate** algebraic expressions if we know the value of the variable(s).*

Example: $Evaluate\ \ 2x-y^2\ \ if\ x=3\ and\ y=2$

$$2(3)-2^2=6-4=2$$

- **Equation**: is an equality of algebraic expressions.

Example:
- $2x^2-x=5x-3$
- $2x+3y=5x-2$ can you give any more examples?

- **Solution/Root**: of an equation is a value for the variable that makes the equation true.

Example: $x=2$ *is a solution to the equation* $5x-3=4x-1$

$$since\ \ 5\cdot 2-3=4\cdot 2-1$$
$$10-3=8-1$$
$$7=7 \downarrow \ \textit{the equality is true}$$

Polynomials

A **Polynomial in *x*** is a finite sum of terms of the form ax^n, where *a* is a real number and *n* is a whole number.

Ex: $x^6 + 9x^5 - 2x^3 + 4x^2 + 5x - 7$, $2x^3 + 5x - 8$

$2x^{-3} + 4x - 3$ is **not a polynomial**

Fun things we do with polynomials

- ***Evaluating:*** Find the value of the polynomial $6x^2 + 11x - 20$ when x = -1.

$$6(-1)^2 + 11(-1) - 20 = 6 - 11 - 20 = -25$$

- ***Simplifying, Adding and Subtracting:*** *We simplify a polynomial by* ***combining like terms***

Example: $14y^2 + 3 - 10y^2 = 4y^2 + 3$

$$(5x^2 - 2x + 1) - (-6x^2 + x - 1) = 5x^2 - 2x + 1 + 6x^2 - x + 1 = 11x^2 - 3x + 2$$

- ***Multiplying:*** *Multiply each term of the first pol. to each term of the second pol. and then combine like terms*

Example:

$$(5y^2 - 6y + 7)(4y + 3) = 5y^2(4y + 3) - 6y(4y + 3) + 7(4y + 3) =$$
$$= 20y^3 + 15y^2 - 24y^2 - 18y + 28y + 21 = 20y^3 - 9y^2 + 10y + 21$$

Special Products

- **FOIL** = **F**irst-**O**uter-**I**nner-**L**ast

$$(x+7)(x-5) = \underbrace{(x)(x)}_{F} + \underbrace{(x)(-5)}_{O} + \underbrace{(7)(x)}_{I} + \underbrace{(7)(-5)}_{L} = x^2 - 5x + 7x - 35 = x^2 + 2x - 35$$

- **Squaring a Binomial:** $\boxed{(a+b)^2 = a^2 + 2ab + b^2}$ *or* $\boxed{(a-b)^2 = a^2 - 2ab + b^2}$

Examples: $(2x+5)^2 = (2x)^2 + 2\cdot(2x)\cdot 5 + (5)^2 = 4x^2 + 20x + 25$

- **Difference of Squares:** $\boxed{(a+b)(a-b) = a^2 - b^2}$

Examples:

$$\left(x-\frac{1}{3}\right)\left(x+\frac{1}{3}\right) = x^2 - \left(\frac{1}{3}\right)^2 = x^2 - \frac{1}{9} \quad or \quad (2x^2+6x)(2x^2-6x) = (2x^2)^2 - (6x)^2 = 4x^4 - 36x^2$$

Dividing Polynomials

- Dividing by a **Monomial**: *Divide each term of the polynomial by the monomial.*

Recall addition of fractions: $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}, \quad c \neq 0$

Ex: $\frac{25x^3 + 5x^2}{5x} = \frac{25x^3}{5x} + \frac{5x^2}{5x} = 5x^2 + x$

- Dividing **by a polynomial other than a monomial**: *Use long division*

Ex: $\frac{x^2+7x+12}{x+3} \to$

$$\begin{array}{r|l} & \quad x+4 \\ \hline x+3 & x^2+7x+12 \\ & -(x^2+3x) \\ \cline{2-2} & \quad 4x+12 \\ & \quad -(4x+12) \\ \cline{2-2} & \qquad 0 \end{array}$$

Factoring

- **GCF-Greatest Common Factor:** When an integer is written as the product of 2 or more integers, each of these is called a factor of the product.

Ex: $\underbrace{2 \cdot 5}_{factors} = \underbrace{10}_{product}$ This is true for polynomials

$$\underbrace{x^2 \cdot x^5}_{factors} = \underbrace{x^7}_{product} \quad or \quad \underbrace{(x+1)(x+4)}_{factors} = \underbrace{x^2 + 5x + 4}_{product}$$

- **GCF of a list of Integers** is the greatest factor that all integers in the list have in common.

To find it, we write each integer in the list as a product of prime factors and then we find the factors that are common to each integer. ***The product of those common factors is the GCF*** *of the integers in that list.*

Example: Find the $GCF(45, 75)$

The prime factorization is

$$\left.\begin{aligned} 45 &= 3 \cdot 3 \cdot 5 \\ 75 &= 3 \cdot 5 \cdot 5 \end{aligned}\right\} \Rightarrow GCF = 3 \cdot 5 = 15$$

24. Practice: Find the GCF of the following numbers:

$32 \text{ and } 33 =$ $\qquad\qquad$ $24, 60 \text{ and } 96 =$

- **GCF of a list of Common Variables** is the variable raised to the smallest exponent in the list

Example: Find the $GCF(x^5, x^3) \rightarrow$

$$\left.\begin{aligned} x^5 &= x \cdot x \cdot x \cdot x \cdot x \\ x^3 &= x \cdot x \cdot x \end{aligned}\right\} \Rightarrow GCF = x \cdot x \cdot x = x^3$$

- **GCF of a list of a list of TERMS is** the product of the GCF of the numerical coefficient times the GCF of the variable factors.

Example:

$$GCF(-9x^2, 15x^4, 6x)\} \Rightarrow \begin{cases} GCF(-9, 15, 6) = 3 \\ GCF(x^2, x^4, x) = x \end{cases} \quad So\ the\ GCF(-9x^2, 15x^4, 6x) = 3x$$

<u>The fist step to factoring a polynomial is to find the GCF of its terms</u>

- **Prime Polynomial** – a polynomial that cannot be factored (cannot be written as the product of two polynomials).

- **Factoring by Grouping** **(used for polynomials with 4 terms)**

Examples: *Factor the polynomials*

$$\underbrace{ab+4a}_{\downarrow}+\underbrace{7b+28}_{\downarrow}=$$

$$a(b+4)+7(b+28)= \leftarrow \textit{the polynomial is not factored yet since the main operation is addition}$$

$$(b+4)(a+7) \qquad \leftarrow \textit{Now the polynomial is completely factored}$$

Practice: *Factor the following polynomials*

25. $5a^2+2ab-5a-2b$

26. $15xz+15yz-5xy-5y^2$

- **Factoring Trinomials of the form** $\boxed{x^2+bx+c}$

By the FOIL method we saw that

$$(x+2)(x+3)=\underset{F}{\overset{\uparrow}{x^2}}+\underset{O}{\overset{\uparrow}{2x}}+\underset{I}{\overset{\uparrow}{3x}}+\underset{L}{\overset{\uparrow}{6}}=x^2+5x+6$$

So $(x+2)(x+3)=x^2+5x+6$ (Sum, Product)

Example $\quad x^2+9x+20=(x+5)(x+4)$

$$\textit{factors of } 20 \begin{cases} 2,10 \rightarrow sum=12 & \textit{not what we are looking for} \\ 4,5 \rightarrow sum=9 & \textit{this is what we want so these are our numbers} \end{cases}$$

Practice: *Factor the following*

27. $x^2-13x+22=$

28. $x^2+5x-36=$

- **Trinomial of the form** $\boxed{ax^2+bx+c}$

In this case we will method *of* ***grouping.***

Step 1. *Find two numbers whose product is* $a\cdot c$ *and sum is* b

Step 2. *Write* bx *as a sum / difference of the above numbers*

Step 3. *Factor by grouping*

Example: $4x^2+12x+5$

We want to find two numbers whose product is going to be $4\cdot 5=20$ *and whose sum is* 12. *Those two numbers are* 2 *and* 10.

$$So\ 4x^2+12x+5=\underbrace{4x^2+2x}_{GCF=2x}+\underbrace{10x+5}_{GCF=5}=2x(2x+1)+5(2x+1)=(2x+1)(2x+5)$$

29. Practice: Factor the following

$3x^2+14x+8=$

$6x^2-7x-5=$

Factoring Binomials

- **Difference of two squares:** $a^2 - b^2 = (a+b)(a-b)$

  Examples: $a^2 - 16 = (a+4)(a-4)$

30. Practice:

$25x^2 - 1 =$ $p^4 - 81 =$

$48x^4 - 3 =$ $c^2 - \frac{9}{25} =$

- $\boxed{a^2 + b^2}$ **is a PRIME Polynomial.**

- **Sum/Difference of two Cubes:** $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
  $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

  Examples: $x^3 - 27 = (x-3)(x^2 + 3x + 9)$

31. Practice: $125x^3 - 1 =$ $16p^3 + 250y^3 =$

Factoring a Polynomial

Step 1. Are there any common factors? If so, factor out the GCF.

Step 2. How many terms are in the polynomial?

a. If there are **TWO** terms decide if one of the following can be applied.
  i. Difference of two squares: $a^2 - b^2 = (a-b)(a+b)$
  ii. Difference of two cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
  iii. Sum of two cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
  iv.
b. If there are **THREE** terms try one of the following.
  i. Perfect Square trinomial: $a^2 + 2ab + b^2 = (a+b)^2$
  ii. If not a perfect square trinomial, factor using the methods presented above
c. If there are **FOUR** or more terms try factoring by grouping.

Step 3. See if factors in the factored polynomial can be factored further.

Rationals

➢ **Rational Expression,** is an expression that can be written in the form $\frac{P}{Q}$ where P and Q are polynomials and $Q \neq 0$.

Domain of a rational expression is the set of real numbers for which the equation is defined.

Since a fraction is not defined when the denominator is zero, in order to find the domain of a rational expression we look for values that make the denominator equal to zero.

The domain is then all real numbers except the values that make the denominator zero.

$$\underline{Ex}: \ \frac{2x-5}{x+3} \textit{ is defined for } \boxed{x \neq -3}, \textit{ or the Domain is } (-\infty,-3)\cup(-3,\infty)$$

Remember: When dealing with Rational Functions always factor first

Example: Find the domain of the following expressions:

$$\frac{4}{x-2}, \qquad \frac{x}{x^2-1}, \qquad \frac{x}{x^2-12x+36}, \qquad \frac{2x-3}{6x^2-5x+1}$$

➢ **Operations**

- **Simplifying**: Always factor first and then cancel like factors.

Practice: Simplify $\dfrac{x^2+6x+5}{x^2-25}$

- **Multiplying**: Multiply numerators together, and denominators together $\frac{p_1}{p_2} \cdot \frac{p_3}{p_4} = \frac{p_1 p_3}{p_2 p_4}$

  Practice: $\frac{x-7}{x-1} \cdot \frac{x^2-1}{3x-21}$

- **Dividing**: Multiply the first fraction by the reciprocal (flipp the second) of the second $\frac{p_1}{p_2} \div \frac{p_3}{p_4} = \frac{p_1}{p_2} \cdot \frac{p_4}{p_3}$

  Practice: $\frac{x^2-2x-8}{x^2-9} \div \frac{x-4}{x+3}$

- **Adding & Subtracting**: Need to find the LCD first and then rewrite them into equivalent fractions with the same denominator $\frac{p_1}{p_2} \pm \frac{p_3}{p_4} = \frac{p_1 p_4 \pm p_3 p_2}{p_2 p_4}$

  Practice: $\frac{x+3}{x^2+x-2} + \frac{2}{x^2-1}$

Complex Fractions

- Multiply every by the LCD of all denominators to simplify.

Simplify: $\dfrac{\dfrac{1}{x}-\dfrac{3}{2}}{\dfrac{1}{x}+\dfrac{3}{4}}$

Simplify: $\dfrac{2d}{\dfrac{d}{r_1}+\dfrac{d}{r_2}}$

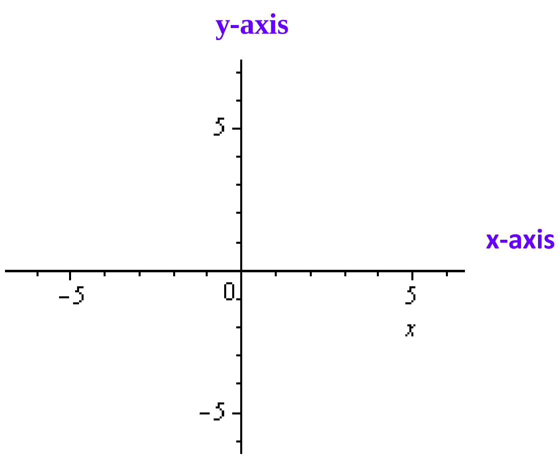
Class 2: Linear Models

Objectives:

- ~ Graph Equations on the Rectangular coordinate system.
- ~ Solve Linear Equations in One Variable
- ~ Solve Rational Equations with Variables on the denominator
- ~ Use Linear Equations to Solve Problems.
- ~ Solve a Formula for a Variable

The Rectangular Coordinate System

Every axis is a Number Line and where they intersect is the origin.



Every point in the coordinate system is an ordered pair (x, y)

and

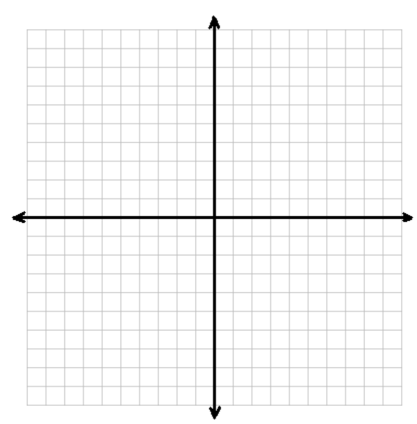
Every ordered pair (x, y) is a point in the coordinate system

The way we plot a point (a,b) in the coordinate system is by starting at the origin.

Then move horizontally (over) a- units then vertically (up) b- units.

Practice 1: Plot the following points in the coordinate system below.

(4, 2), (2, -2), (-1, -3), (-5, 1), (0, 2), (3, 0), (0, -4), (-4, 0)

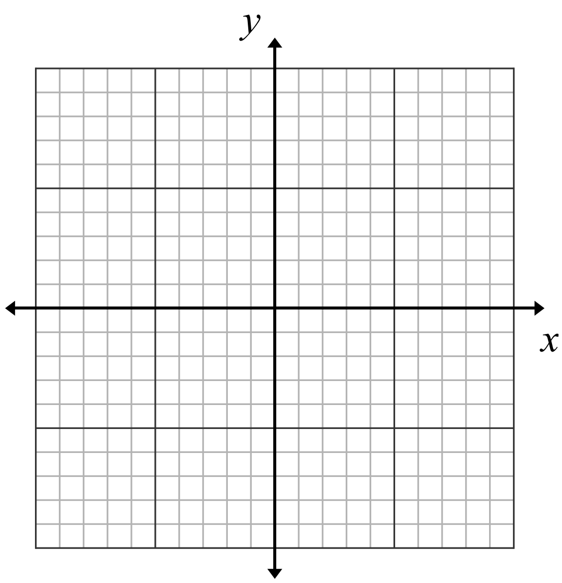


- An equation in two variables such as $2x - y = 1$ or $y = 4x^2 + 3$ has a solution consisting of two values, (x, y). So an equation in two variables has as a solution an ordered pair (x, y).

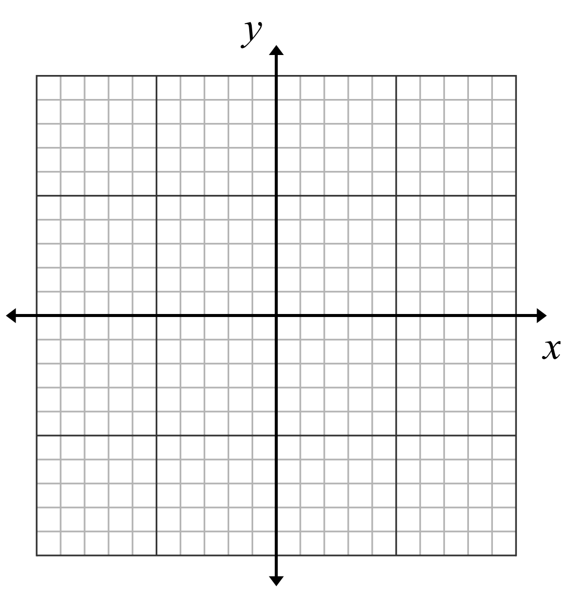
- An ordered pair is a solution of an equation in two variables if replacing the values for the variable makes the equation true.

The graph of an equation in two variables is the set of all points whose coordinates satisfy the equation.

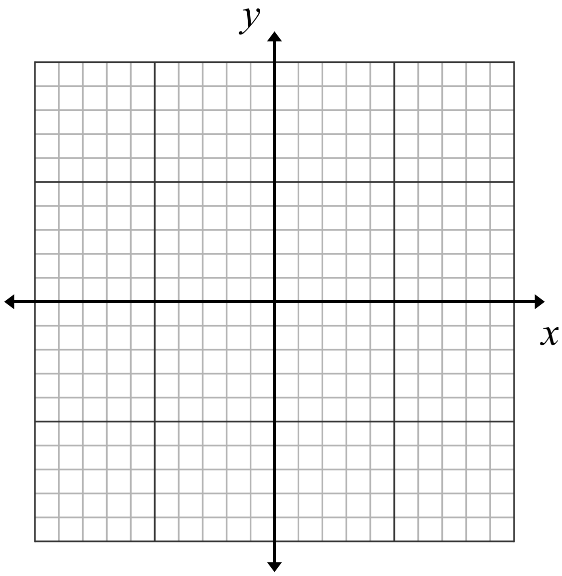
Practice 2: Graph the equation $y = 2x - 3$ by using the point-plotting method.



Practice 3: Graph the equation $y = x^2 + 3$ by using the point-plotting method.



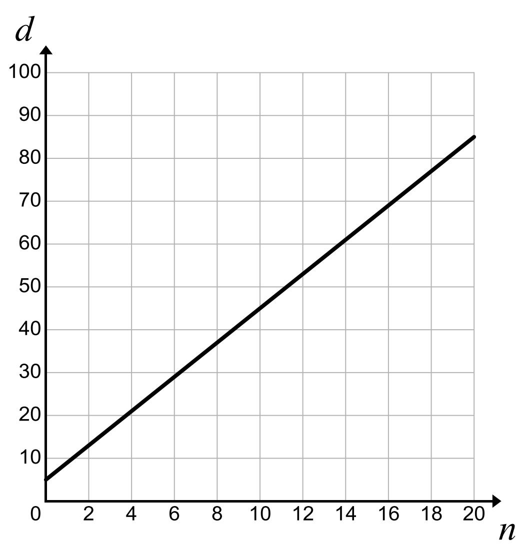
Practice 4: Graph the equation $y = |x+3| - 2$ by using the point-plotting method.



- **Intercepts**
  - **X-Intercept** is the point where the graph crosses the x-axis.
    - We find it by plugging 0 for y and solving the equation for x.
  - **Y -Intercept** is the point where the graph crosses the y-axis.
    - We find it by plugging 0 for x and solving the equation for y.

- **Interpreting Information given by Graphs:**

Practice 5: Divorce rates are considerably higher for couples who marry in their teens. The graph below shows the percentage of marriages ending in divorce based on wife's age at marriage.



Determine the percentage of marriages ending in divorce after 15 years.

Solving Equations

General Strategy of Solving Linear Equations

1. Multiply both sides by the LCD to clear fractions
2. Use the Distributive Property to remove parenthesis
3. Simplify each side by combining like terms
4. Get all the variables in one side of the equation and all the number to the other
5. Get the variable alone by using the Multiplicative Property of equality
6. Check the solution by substituting it in the equation

Practice: Solve each of the equations

6. $-6(2x+1)-14=-10(x+2)-2x$

7. $\frac{3(x-2)}{5}=3x+6$

8. $\frac{2}{a}-\frac{1}{x}=\frac{3}{b}$

9. $\frac{a(x-c)}{b}-2x=\frac{5c}{3}$ for x

10. $\dfrac{x-5}{x-3}+2=2x-7$

11. $\dfrac{2t+3}{t-1}-\dfrac{2}{t+3}=\dfrac{5-6t}{t^2+2t-3}$

Literal Equations

Solve each of the following equations:

12. $C=\dfrac{5}{9}(F-32)$

13. $V=C-\dfrac{C-S}{L}N$

14. In Electrochemistry, voltaic cells are unlikely to be operated under standard state conditions. The voltage can still be calculated by the use of the Nernst equation $E=\dfrac{2nFG-2.30RM\,(C+273)}{nF}$. Solve for F.

Word Problems

General Strategy of Solving Word Problems
1. **Understand** the problem: read and reread and choose a variable for the unknown
2. **Translate** the problem into an equation
3. **Solve** the equation
4. **Interpret** the result: check and state your conclusion

Example:

1. Twice the difference of a number and 8 is equal to three times the sum of the number and 3. Find the number.

- **Understand**: Read the problem $\Rightarrow$ x = the unknown number
- **Translate**:

Twice		Difference of a # and 8	Is equal to	3 times		Sum of the # and 3
2	**•**	**(x-8)**	**=**	**3**	**•**	**(x+3)**

- **Solve**: $2(x-8)=3(x+3)$
- **Interpret**: Check you result. State: The number is ____________

2. To make an international call, you need the code for the country you are calling. The codes for Belgium, France and Spain are three consecutive integers whose sum is 99. Find the code for each country.

- **Understand**: Read the problem $\Rightarrow$ x = the code for Belgium
  x+1 = the code for France
  x+2 = the code for Spain

- **Translate**:

Belgium	sum	France	Sum	Spain	equals	99
x	**+**	**(x+1)**	**+**	**(x+2)**	**=**	**99**

- **Solve**: $x+(x+1)+(x+2)=99$
- **Interpret**: Check you result. State: The number is ____________

Practice:

15. The sum of twice a number and 7 is equal to the sum of a number and 6.

16. If ¾ is added to three times a number, the result is ½ subtracted from twice the number.

17. The room numbers of two adjacent classrooms are two consecutive even numbers. If their sum is 654, find the classroom numbers

18. A 40-inch board is to be cut into three piece so that the second piece is twice as long as the first piece and the third piece is 5 times as long as the first piece. Find the lengths of all three pieces.

Formulas

$A = lw$	Area of a Rectangle $=$ length$\times$width
$P = 2l + 2w$	Perimeter of a Rectangle $=$ Sum of lengths of all sides
$P = a + b + c$	Perimeter of a Triangle $=$ Sum of lengths of all sides
$A = \frac{1}{2}bh$	Area of a Triangle $= \frac{1}{2} \cdot$ base $\cdot$ height
$V = lwh$	Volume $=$ lenght $\cdot$ width $\cdot$ height
$A = \pi r^2$	Area of a Circle $= \pi \cdot$ radius
$P = 2\pi r$	Perimeter of a Circle $= 2 \cdot \pi \cdot$ radius
$d = rt$	Distance $=$ rate $\cdot$ time
$I = PRT$	Interest $=$ Principal $\cdot$ rate $\cdot$ time
$F = \frac{9}{5}C + 32$	Degrees Fahrenheit $= \frac{9}{5} \cdot$ Degrees Celcius$+32$

19. A car rental agency advertised renting a Buick Century for $24.95 per day and $0.29 per mile. If you rented this car for 2 days, how many whole miles can you drive on a $100 budget?

20. A family is planning their vacation to Disney World. They will drive from a small town outside Atlanta, Georgia, to Orlando, Florida, a distance of 350 miles. They plan to average a rate of 55mph. How long will this trip take?

Further Problem Solving

- **Solve problems involving Percents**

$$\boxed{Increase = Current - Original}$$

$$\boxed{Decrease = Original - Current}$$

$$\boxed{Percent\ Increase = \frac{Increase}{Original}}$$

$$\boxed{Percent\ Decrease = \frac{Decrease}{Original}}$$

- **Solve problems involving Distance** $\boxed{D = r \cdot t}$
- **Solve problems involving Interest** $\boxed{I = PRT}$

21. Nordstrom's advertised a 25% off sale. If a London Fog coat originally sold for \$256, find the decrease in price and the sale price

22. Find the original price of a pair of shoes if the sale price is \$78 after a 25% discount.

23. How many cubic centimeters (cc) of a 25% antibiotic solution should be added to 10cc of a 60% antibiotic solution to get a 30% antibiotic solution?

24. Planters Peanut Company wants to mix 20 pounds of peanuts worth \$3.00 a pound with cashews worth \$5.00 a pound in order to make an experimental mix worth \$3.50 a pound. How many pounds of cashews would be added to the peanuts?

25. A jet plane traveling at 500mph overtakes a propeller plane traveling at 200mph that had a 2-hour head start. How far from the starting point are the planes?

26. Karen invested some money at 9% annual simple interest and $250 more than that amount, at 10% annual simple interest. If her total yearly interest was $101, how much was invested in each?

27. How can $54,000 be invested, part at 8% annual simple interest and the remainder at 10% annual simple interest, so that the interest earned by the two accounts will be equal?

28. A bus traveled on level road for 3 hours at an average speed of 20mph faster than it traveled on a winding road. The time spend on the winding road was 4 hours. Find the average speed on the level road if the entire trip was 305 miles.

29. How much pure acid should be mixed with 2 gallons of a 40% acid solution in order to get a 70% acid?

30. Find the original price of a pair of shoes if the sale price is $78 after a 25% discount.

Class 3: Quadratics

Objectives:

- ~ Perform Operations with Complex Numbers
- ~ Solve Quadratic Equations by any method
- ~ Solve Polynomial Equations by factoring
- ~ Solve Radical Equations.
- ~ Solve Equations with Rational Exponents
- ~ Solve Equations involving Absolute Values
- ~ Solve linear and Absolute Value Inequalities

- **The Imaginary Unit is**____

- **A complex Number is**____

  ____

- **Complex Conjugate**____

➢ Operations with Complex Numbers

○ Powers of Imaginary Numbers.

Practice 1. Perform the indicated Operation

$(i)^5 =$

$(i)^{33} =$

$(5i)^4 =$

$(-2i)^7 =$

- **Addition/Subtraction:** Add/Subtract the real parts together and the imaginary parts together.

  Practice 2. $(-7+5i)-(-9-11i)=$

- **Multiplication.** Same as multiplication of polynomials, use FOIL

  Practice 3: $(9-45i)(5-i)$

  Practice 4: $(2+7i)(2-7i)=$

- **Division.** Multiply numerator and denominator with the conjugate of the denominator and simplify.

  Practice 5: $\dfrac{8+5i}{8-5i}$

Quadratic Equation

Quadratic Equation - is an equation that can be written in the form $ax^2+bx+c=0$ where *a*, *b* and *c* are real numbers and $a \neq 0$.

Examples: $3x^2-2x+7=0, \quad x^2=1, \quad y^2+y=4$

A quadratic Equation in x is also called a **Second Degree Polynomial Equation** in x.

- **Methods to Solve Quadratic Equations:**

  Recall: *On chapter 6 we learned how to solve quadratic equations by factoring. In order to solve a quadratic equation by factoring we use the* ***Zero Factor Theorem***.

  - **Zero Factor Theorem**: If *a* and *b* are real numbers and if *ab* = 0 then *a* = 0 or *b*= 0.

Example: $x^2-5x-14=0 \Rightarrow \underbrace{(x-7)}_{a}\underbrace{(x+2)}_{b}=0 \Rightarrow (x-7)=0 \quad or \quad (x+2)=0$

$$x=7 \quad or \quad x=-2$$

- **Factoring :**

Solving Quadratic Equations by Factoring:

1. ***Write the eq. in standard form.*** $ax^2+bx+c=0$
2. ***Factor the quadratic completely.***
3. ***Set each factor containing a variable equal to zero.***
4. ***Solve the resulting equation.***
5. ***Check each solution in the original equation.***

Example: *Solve the equation*

$$x(x-4)=5 \Rightarrow \quad Note: We\ \underline{cannot}\ set\ x=5\ and\ x-4=5$$

$$x(x-4)-5=0 \Rightarrow \quad So\ we\ put\ the\ equation\ in\ standard\ form$$

$$x^2-4x-5=0 \Rightarrow \quad by\ bringing\ everything\ one\ one\ side$$

$$(x-5)(x+1)=0 \Rightarrow \quad check! \rightarrow 5(5-4)=5\cdot 1=5\surd$$

$$x-5=0 \quad x+1=0 \quad -1(-1-4)=-1\cdot -5=5\surd$$

$$x=5 \quad x=-1$$

Practice:

6. $-5x^2+20x+60=0$

7. $2x^3-18x=0$

8. $(x+3)(3x^2-20x-7)=0$

9. $3x^3-9x^2-12x=0$

- **Square Root Property** $\boxed{\text{If } x^2=a \quad for \;\; a\geq 0 \Rightarrow x=\pm\sqrt{a}}$

Example: $x^2-49=0 \Rightarrow x^2=49 \Rightarrow x=\pm\sqrt{49} \Rightarrow x=\pm 7$

Practice: Solve the following using the square root property.

10. $3x^2=11$

11. $(x-4)^2=36$

12. $(4x+1)^2=15$

13. $(x+3)^2=-5$

- **Completing the Square**

To complete the Square on $\boxed{x^2 + bx}$

- **Add** $\left(\frac{b}{2}\right)^2$ **to both sides.**
- **Find** $\left(\frac{b}{2}\right)^2$**, find half the coefficient of** x**, square it, and add it to both sides of the equation**

Example

$x^2+8x+1=0 \Rightarrow$ *put in the standard form*

$\underline{\quad -1 \quad -1}$

$x^2+8x=-1 \Rightarrow$ *find b, the coefficient of x*

$x^2+8x=-1 \Rightarrow$ *Since* $b=8$ *we add* $\left(\frac{8}{2}\right)^2=16$

$\underline{\quad +16 \quad +16}$ *to both sides*

$\underbrace{x^2+8x+16}_{\downarrow}=-1+16 \Rightarrow$ *write it as a perfect square*

$(x+4)^2=15$

Solving a Quadratic Eq. in x by Completing the Square

Step 1: If the coefficient of x^2 is 1, go to Step 2. If not, divide both sides of the eq. by that coefficient

Step 2: Get all terms with variables on one side of the eq. and constants on the other side

Step 3: Find half the coefficient of x and then square the result. Add this number to **both** sides of the eq.

Step 4: Factor the resulting perfect square trinomial

Step 5: Use the square root property to solve the eq.

Example: $x^2+2x=4 \Rightarrow$ *put it in the form* ax^2+b

$\underline{\quad +1 \quad +1} \Rightarrow$

$\underbrace{x^2+2x+1}_{\downarrow}=4+1 \Rightarrow$ *write it as a perfect square*

$(x+1)^2=5 \Rightarrow$ *use the Sq. Rt. property to solve*

$x+1=\pm\sqrt{5} \Rightarrow \boxed{x=-1\pm\sqrt{5}}$

Practice: Solve the equations by Completing the square

14. $y^2+14y=-32$

15. $4x^2-16x-9=0$

16. $2x^2+10x=-13$

17. $2x^2=-3x+1$

Quadratic Formula

Solving a Quadratic Eq. in x by the Quadratic Formula

Step 1: Write the quadratic eq. in standard form $ax^2 + bx + c = 0$

Step 2: If necessary, clear the equation of fractions to simplify calculations

Step 3: Identify *a*, *b* and *c*

Step 4: Replace *a*, *b* and *c* in the quadratic formula with the identified values and simplify

Example:

$$2x^2 - x - 5 = 0 \Rightarrow x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-5)}}{2 \cdot 2} = \frac{1 \pm \sqrt{1+40}}{4} \Rightarrow x = \frac{1 \pm \sqrt{41}}{4}$$

<u>Practice</u>: Solve the equations

18. $3x^2 + 8x = 3$

19. $5x^2 = 2$

20. $x^2 = -2x - 3$

21. $\frac{1}{3}x^2 - x = 1$

- **<u>The Discriminant</u>**
  - **No Solutions if**
  - **1 Solution if**
  - **2 Solutions if**

<u>Practice</u>: Use the discriminant to find the # of solutions

22. $5x^2 + 2x - 3 = 0$

23. $x^2 + 2x + 2 = 0$

24. $x^2 + 2x + 1 = 0$

Other Equations

- **Radical Equation**: an equation in which the variable is in a square root, cubic root or a higher order root.

$$Ex: \ \sqrt{x}=9, \ \sqrt{2x-5}=12, \ \sqrt[3]{x-1}=3 \ \ etc.,$$

Domain: all the x-values (that we can plug in and make the expression defined).

Since a square (even power) root is not defined for a negative radicand, in order to find the domain of a radical expression we set the radicand greater than or equal to zero and solve for the variable.

That will give us the domain of that equation.

$$Ex: \ \sqrt{x-4}=9 \Rightarrow \quad x-4\geq 0 \Rightarrow \boxed{x\geq 4}$$ *So the domain is* $[4,\infty)$

	Solving Radical Equations containing nth Roots
1.	If necessary, arrange all terms so that one radical is isolated on one side of the equation.
2.	Raise both sides of the equation to the nth power to eliminate the nth root.
3.	Solve the resulting equation. If this equation still contains radicals, repeat step 1 & 2
4.	Check all proposed solutions in the original equation

Example: *Solve the equation* $15+\sqrt{3x+17}=x$

$15+\sqrt{3x+17}=x$ *subtract 15 from both sides*

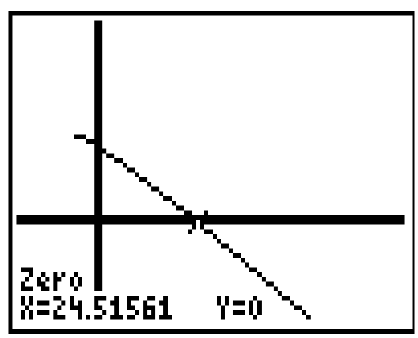
$\sqrt{3x+17}=x-15$ *square both sides*

$3x+17=(x-15)^2$

$3x+17=x^2-30x+225$ *move everything to one side*

$x^2-33x+208=0$

And by graphing:



By using the quadratic formula

$$x=\frac{-(-33)\pm\sqrt{33^2-4(1)(208)}}{2}\Rightarrow$$

$$x=\frac{33\pm\sqrt{257}}{2}=\begin{cases}24.5156 \ \surd \\ 8.4844\end{cases}$$ *by checking the solutions*

<u>Always check solutions</u> by pluging them back in when solving radical equations because of extraneous solutions.

Solving Radical Equations

Step 1: If necessary, arrange terms so that one radical is isolated on one side of the equation.

Step 2: Raise both sides of the equation to the *n*th power to eliminate the isolated *n*th root.

Step 3: Solve the resulting equation. If this equation contains radicals repeat steps 1 and 2.

Step 4: Check all solutions in the original equation.

Practice: *Solve the following equations for their real solutions.*

25. $\sqrt{4x-3}=7$

26. $\sqrt{2x+3}-2x=x-2$

27. $\sqrt{2x-3}-\sqrt{x-2}=1$

Equations with Rational Exponents:

We know that expressions with rational exponents represent radicals.

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

Solving Equations with Rational Exponents:

Assume that m and n are positive integers, m/n is in lowest terms, and k is a real number.

Step 1: Isolate the expression with the rational exponent.

Step 2: Raise both sides of the equation to the n/m power.

Step 3: Solve the resulting equation.

Step 4: Check all solutions in the original equation.

Practice: *Solve the following equations for their real solutions.*

28. $x^{\frac{3}{2}} = 27$

29. $(x+5)^{\frac{3}{2}} = 8$

30. $8x^{\frac{5}{3}} - 24 = 0$

Polynomial Equations and Equations Quadratic in Form

To solve polynomial Equations we use factoring and for Equations Quadratic in Form we use substitution.

Practice: Solve the following equations

31. $4x^3 - 12x^2 = 9x - 27$

32. $3x^4 = 81x$

33. $2x^{\frac{2}{3}} + 4x^{\frac{1}{3}} = 6$

34. $x - 3 = \frac{4}{x}$

Equations Involving Absolute Values

We have seen that the absolute value of x denoted by |x| is the distance of x from zero.

Note: $|x| = \begin{cases} x & if\ x > 0 \\ -x & if\ x < 0 \end{cases}$

Practice: Solve the following Equations

35. $|3x-7|+6=21$

36. $2|3x-2|=14$

37. $|x+1|+6=2$

Inequalities

- **Linear Inequalities:** $mx + b \leq 0$ *or* $mx + b \geq 0$

Practice: Solve the following inequalities

38. $5x + 11 < 26$

39. $-6 \leq \frac{1}{2}x + 4 < -3$

Absolute Value Inequalities

If u is an algebraic expression and c is a positive number, then

$$|u| < c \Leftrightarrow \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

And

$$|u| > c \Leftrightarrow \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

These rules are true if $<$ is replaced by $\leq$ and if $>$ is replaced by $\geq$.

Practice: Solve the following inequalities

40. $|3x+5| < 17$

41. $7|2x-8|+14 \geq 2x+5$

Class 4: Function and Their Graphs

Objectives:

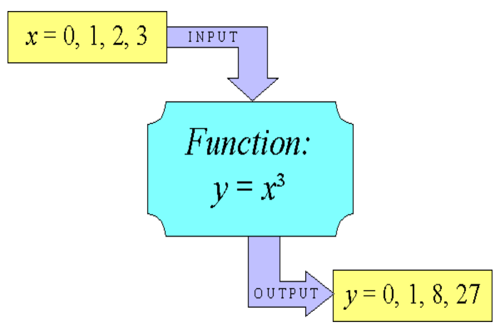
- ~ Identify and Graph Functions
- ~ Identify Domain and Range
- ~ Identify Characteristics of Functions
- ~ Calculate the Difference Quotient of a Function
- ~ Calculate the slope of a Line.
- ~ Write and find the point-Slope and Slope intercept of the equation of a line
- ~ Solve Equations involving Absolute Values

- **A Relation is** ______________________________

- **Domain** ______________________________

- **Range** ______________________________

- **Functions is** ______________________________



There are four possible ways to represent a function:

- **verbally** **(by a description in words)**
- **numerically** **(by a set of ordered pairs)**
- **visually** **(by a graph)**
- **algebraically** **(by an explicit formula)**

Example:

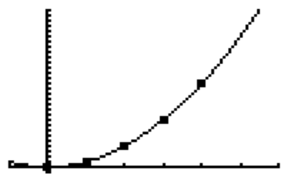
- **Verbally:** The area of a square plot of land is equal to the square of the length of the lot.

- **Numerically:** (0,0), (1,1), (2,4), (3,9), (4,16) ...

Or

Length	**0**	**1**	**2**	**3**	**4**	**...**
Area	0	1	4	9	16	...

- **Visually:**



- **Algebraically:** $A(s) = s^2$

- **Notation:**

A function f of x is represented as: ______________________

x – represents______________________

y - represents______________________

The Graph of a Function ______________________

Determining whether a relation is a function

Numerically

Practice: Determine if the following examples are functions. If not, explain.

1. In the following ordered pairs the first element represents "Number of hours worked" and the second element represents "Total pay".

(0, $0)	(1, $7.50)	(2, $15.00)
(3, $22.50)	(4, $30.00)	(5, $37.50)
(6, $45.00)	(7, $52.50)	(8, $60.00)

2. The first element of each ordered pair is "Student First Name" and the second element of each ordered pair is "Number of Math Courses Taken".

1. (Peter, 2)	2. (Jackie, 0)	3. (Marian, 2)
4. (Tammy, 3)	5. (Jess, 1)	6. (Jackie, 1)
7. (John, 3)	8. (Joe, 2)	9. (Ron, 0)

- **Algebraically**

  Solve the equation in terms of x. If two values of y can be obtained for a given x, the equation is not a function.

Practice: Determine if the following equations define y as a function of x.

3. $x^3 + y = 14$ 4. $x^3 + y^2 = 14$ 5. $xy + 3y = 4$

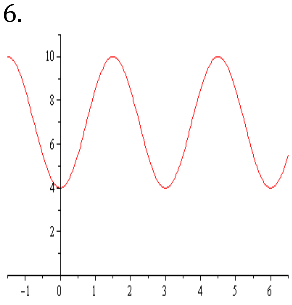
- **Visually**

- **The Vertical Line Test :** ____

____

Practice: Determine if y is a function of x.

6.



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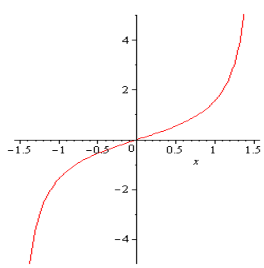

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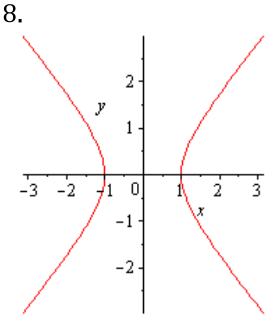


Graph of a Function

- **Arrows indicate**____

- **A closed dot ● indicates** ____________________________________

- **An open dot ○, indicates** ___________________________________

➢ Finding Domain and Range

Practice: Find the Domain and Range in each of the following cases:

○ Numerically

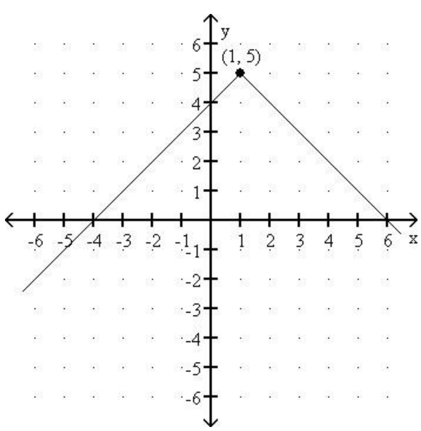
6. (0, 1650), (10, 1750), (20, 1860), (30, 2070), (40, 2300),
(50, 2560), (60, 3040), (70, 3710), (80, 4450), (90, 5280)

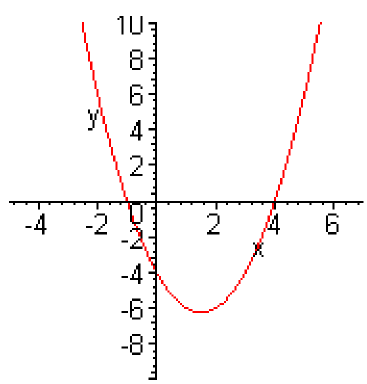
Domain ____

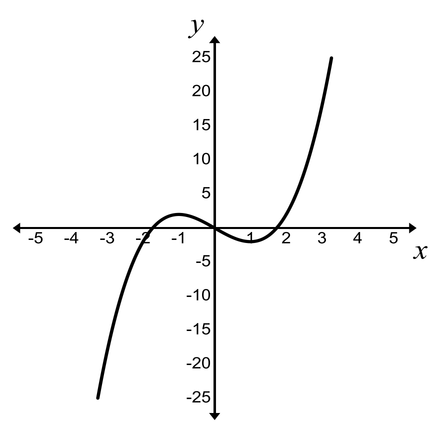
Range ____

○ Visually

7.







Domain: ________________ ________________ ________________

Range: ________________ ________________ ________________

http://lima.osu.edu/people/iboyadzhiev/GeoGebra/domain_range.html

- **Algebraically**

  The Domain of any polynomial function is all real numbers (-∞, ∞) and we can determine the range by graphing it.

  Exceptions:

  - When the function contains **a fraction**, then we need to exclude the values that make the denominator equal to 0.
  - When the function contains **square (or even) roots**, we need to exclude the values that would make the radicand negative.

Evaluating Functions

Same process as evaluating an algebraic expression

8. Example: Consider the function $f(x) = 2x^2 - 5x + 3$. Evaluate the following:

a. $f(-3)$

b. $f(h)$

c. $f(h+2)$

Difference Quotient

Difference Quotient at a general point is found by using the formula. $\boxed{\dfrac{f(x+h)-f(x)}{h}, \quad h \neq 0}$

It is used to find a formula for the rate of change of a quantity (Derivative).

Steps:

1. *Create f(x+h)*
2. *Substitute f(x+h) and f(x) into the equation*
3. *Distribute the negative sign and combine like terms*
4. *Factor out common factor (h) from the top and reduce with h on the denominator*

Example: Calculate the difference quotient for the function $f(x) = 2x^2 - 7x - 11$

$$\left.\begin{array}{l} f(x+h) = 2(x+h)^2 - 7(x+h) - 11 = \\ 2(x^2 + 2xh + h^2) - 7(x+h) - 11 = \\ 2x^2 + 4xh + 2h^2 - 7x - 7h - 11 \end{array}\right\} \textit{step } 1$$

$$\left.\frac{2x^2 + 4xh + 2h^2 - 7x - 7h - 11 - (2x^2 - 7x - 11)}{h}\right\} \textit{step } 2$$

$$\left.\frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{-7x} - 7h \cancel{-11} \cancel{-2x^2} \cancel{+7x} \cancel{+11}}{h} = \frac{4xh + 2h^2 - 7h}{h} = \right\} \textit{step } 3$$

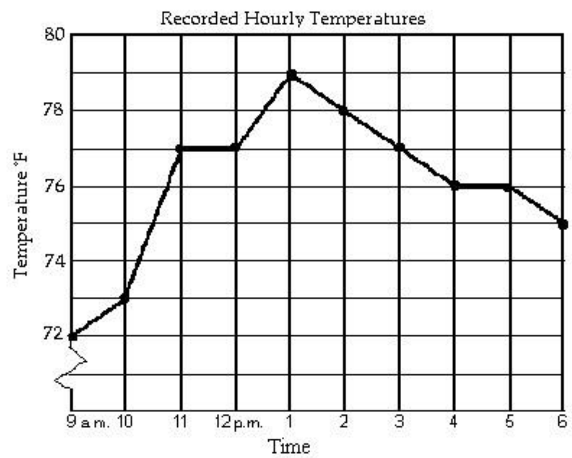
$$\left.\frac{\cancel{h}(4x + 2h - 7)}{\cancel{h}} = \right\} \textit{step } 4 \quad \Rightarrow \quad \boxed{4x + 2h - 7}$$

9. Find the Difference Quotient for $f(x) = 2x^2 - 5x + 3$

Characteristics of Functions:

- **DOMAIN** - all the x-values
- **RANGE** - all the y-vales
- **MAX/MIN** - the highest/lowest point of the graph in an interval
- **Increase/Decrease** - intervals where the function is increasing or decreasing
- **X-INTERCEPTS** - the point where the graph crosses the x-axis
- **Y-INTERCEPTS** - the point where the graph crosses the y-axis
- **Odd** – the graph is symmetric with respect to the origin. $f(-x) = -f(x)$
- **Even-** the graph is symmetric with respect to the y-axis. $f(-x) = f(x)$

10. The line graph shows the recorded hourly temperatures in degrees Fahrenheit at an airport. Use the graph to answer the questions.



a. When is the temperature increasing?

____

b. When is the temperature constant?

____

c. When is the temperature the highest?

____

d. What temperature was recorded at 3 pm?

____

e. During which time period the temperature recorded was at least 77°?

____

f. During which time was the change in temperature the greatest?

____

g. State the Domain and Range of this function______________________________

Piece-Wise Functions

Piece-wise functions are functions that are graphed in pieces

12. ***Practice*:** Graph the following function.

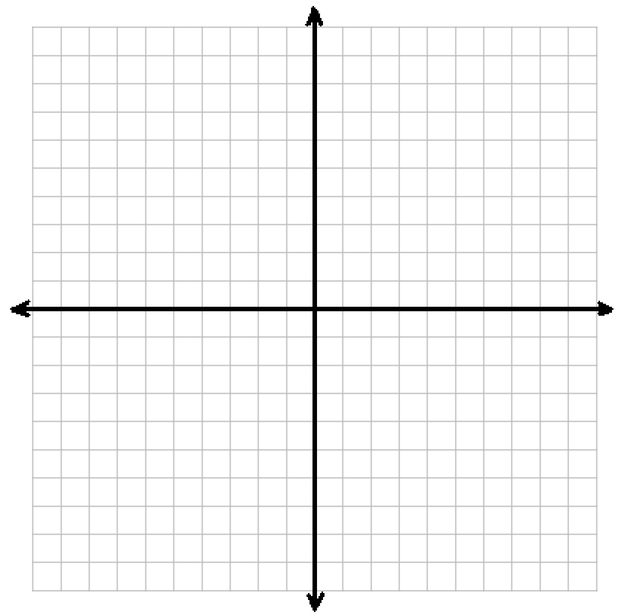
$$f(x)=\begin{cases} x-3, & if\ \ x<0 \\ 4, & if\ \ x=0 \\ x+6, & if\ \ x>0 \end{cases}$$

Domain:

Range:

Evaluate f(-2)=

f(3)=



Linear Functions and Slope

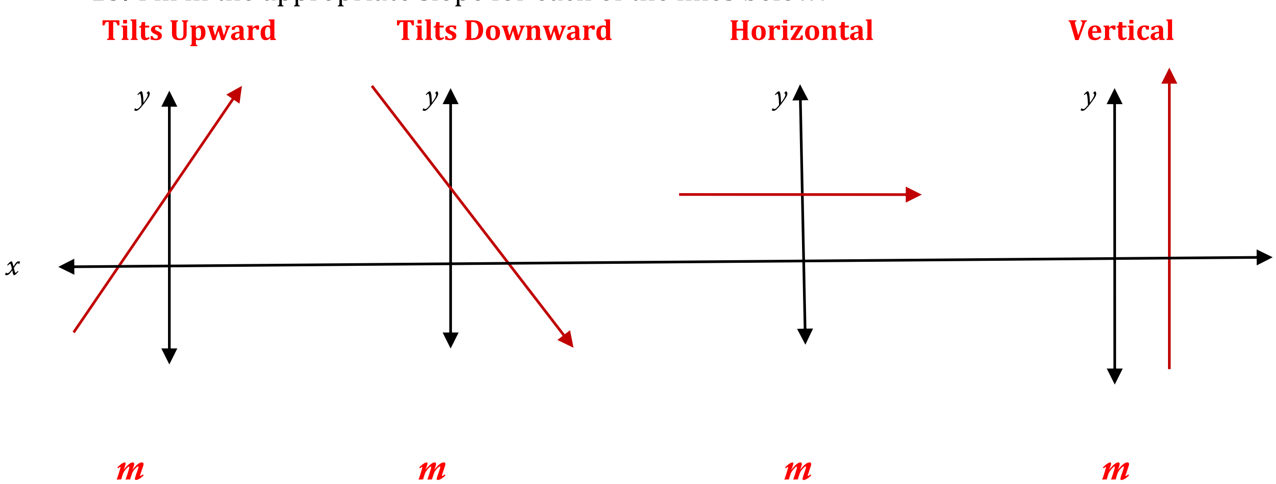
- **Write the General Form of the Equation of a Line: ______________________________**
- **Write the equation of a horizontal line:______________________________**
- **Write the equation of a vertical line:______________________________**
- **Slope is______________________________**
- **Write the formula used to find the slope of a line ______________________________**
- **Write the slope -intercept form of a linear equation and state what each part represent.**

  ____
- **Write the Point-Slope form of the Equation of Line: ______________________________**
- *State the appropriate slope for each of the following cases:*

  Vertical Line ______________________________

  Horizontal Line______________________________

13. Fill in the appropriate slope for each of the lines below:



Practice: For the each of the following find the slope of the line through the points:

14. (-2,-5) ,(0, -2), (4,4), (10, 13)

15. (-2,1), (3,5)

16. State the slope of each of the lines given by the equations below:

a. $y = 3x - 5$ b. $y = -\frac{x}{7} + 4$

17. Find the equation of the line that goes through the points (-2,3) and (-5, -1).

Class 5: Average Rate of Change and Transformations

Objectives:

- ~ Calculate Average Rate of Change
- ~ Recognize Graphs of Common Functions
- ~ Use transformations to graph Functions

- **Average Rate of Change**: $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$

Practice: For each of the following functions, find the average rate of change.

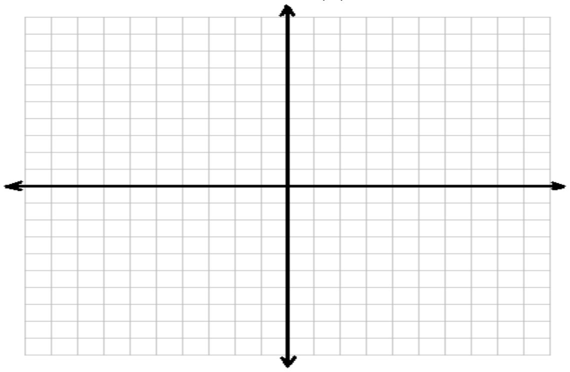
1. $f(x)=6x \quad x_1=0 \ \ to\ x_2=4$

2. $f(x)=x^2-2x \quad x_1=3 \ \ to\ x_2=6$

3. $f(x)=\sqrt{x} \quad x_1=9 \ \ to \ \ x_2=16$

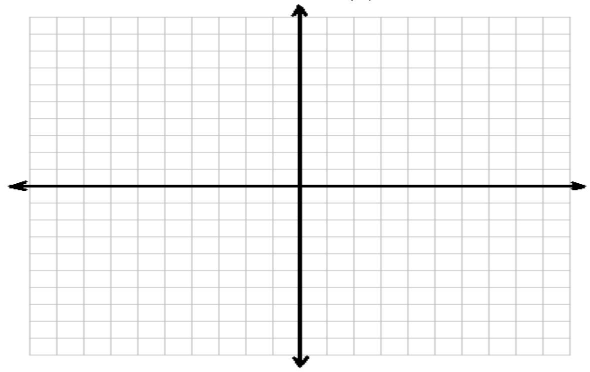
Basic Functions

- Linear f(x)=



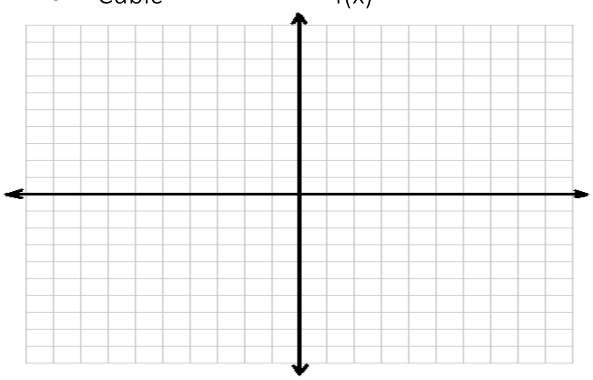
Domain:
Range:

- Quadratic f(x)=



Domain:
Range:

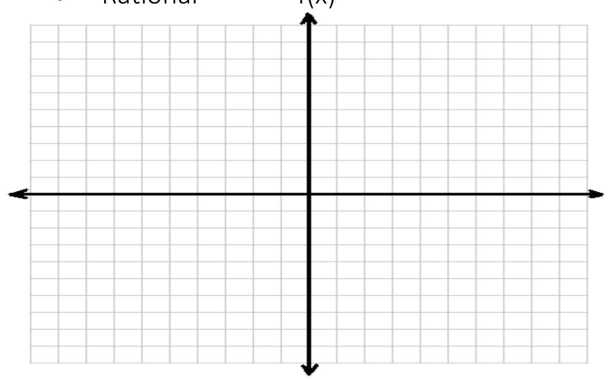
- Cubic f(x)=



Domain:

Range:

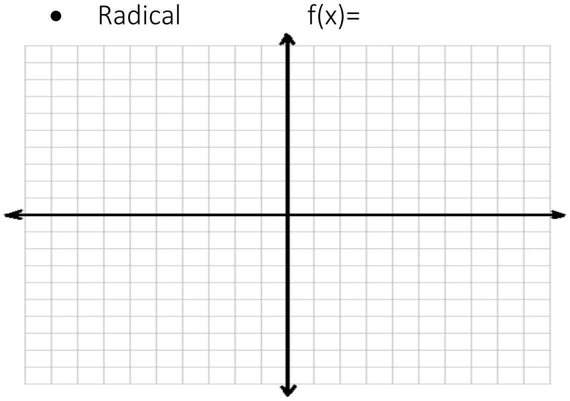
- Rational f(x)=



Domain:

Range:

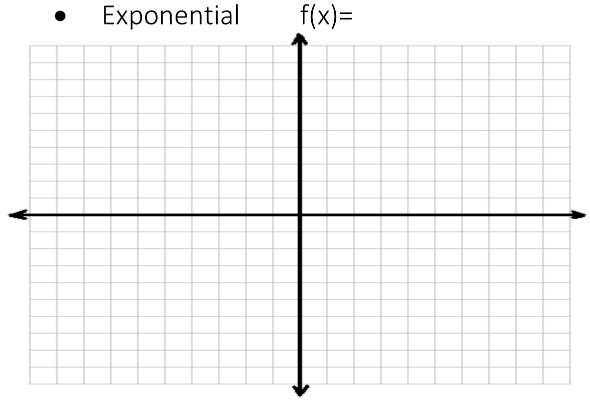
- Radical f(x)=



Domain:

Range:

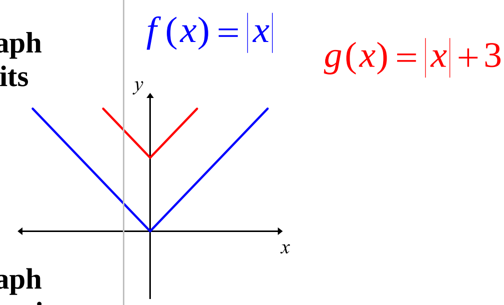
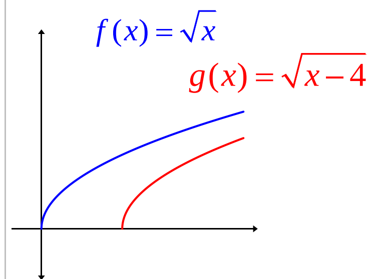
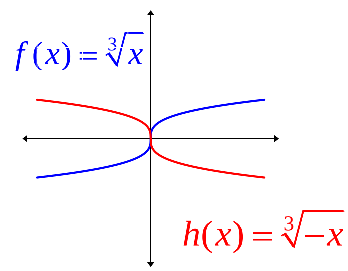
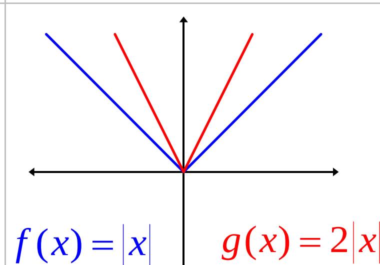
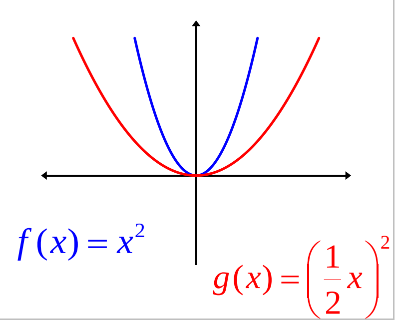
- Exponential f(x)=



Domain:

Range:

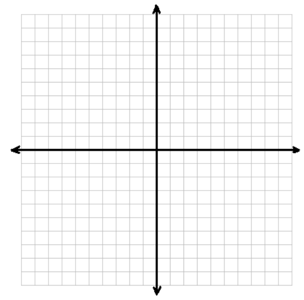
Transformations of Functions

Transformation	Equation	Description	
Vertical translation	$y = f(x) + c$ $y = f(x) - c$	**Shifts the graph upward c units** **Shifts the graph downward c units**	
Horizontal translation	$y = f(x + c)$ $y = f(x - c)$	**Shifts the graph to the left c units** **Shifts the graph to the right c units**	
Reflections	$y = -f(x)$ $y = f(-x)$	**Reflects the graph about the x-axis** **Reflects the graph about the y-axis**	
Vertical Stretching/ Shrinking	$y = cf(x)$	**Vertically stretch if c >1 and** **Vertically Shrinks if 0 < c <1**	
Horizontal Stretching/Shrinking	$y = f(c\,x)$	**Horizontally Shrinks if c >1 and** **Horizontally Stretches if 0 < c < 1**	

Practice: Describe the change in the graph of the function $f(x)=x^2$ for each of the following transformation, and then graph it.

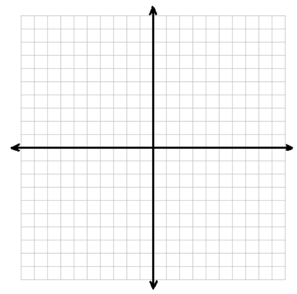
a. $f(x)=x^2+2$ __________________________________

b. $f(x)=2x^2$ __________________________________



c. $f(x)=-x^2$ __________________________________

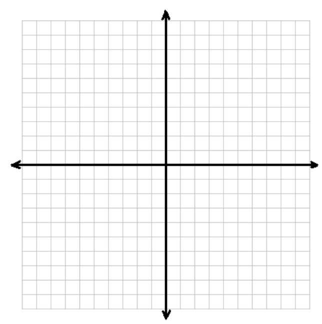
d. $f(x)=(x+2)^2$ __________________________________



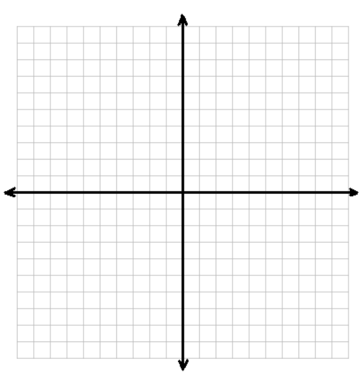
e. $f(x)=\left(\frac{1}{2}x\right)^2$ __________________________________

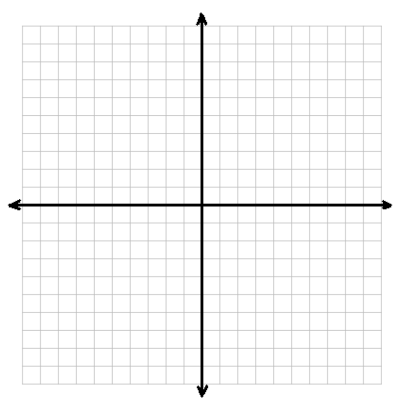
f. $f(x)=-(2x+6)^2-5$

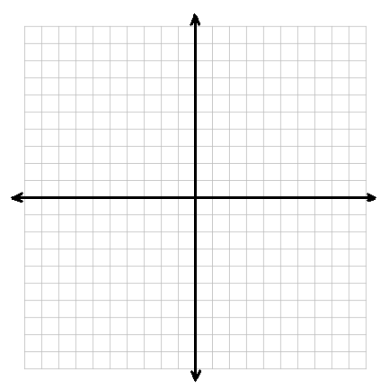
____



g. Is this function odd or even?____

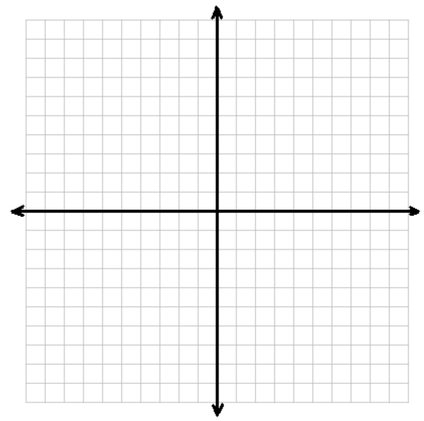




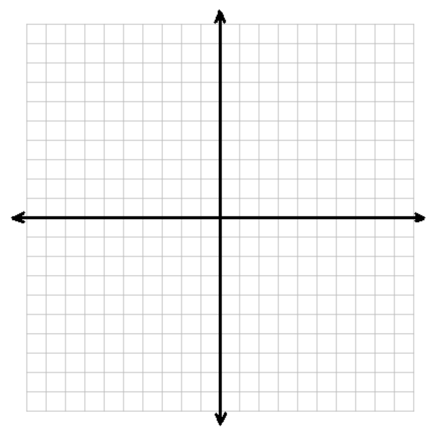


Practice: Sketch the graph of the following function and write out the transformations:

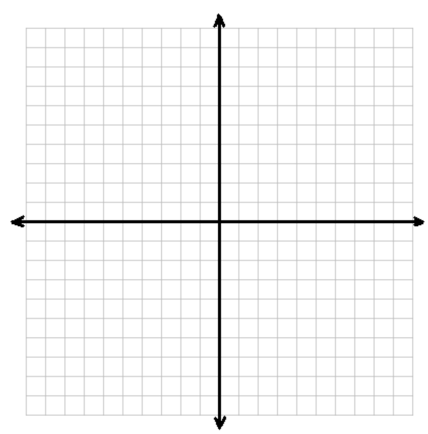
4. $f(x) = (x-5)^3 - 4$



5. $g(x) = -\sqrt{x+4} + 3$



6. $j(x) = 2^{(x+3)} - 6$



Class 6: Composition and Inverse Functions

Objectives:

~ Combine functions using the algebra of functions
~ Determine domain of Functions and of composite functions
~ Write Functions as Compositions
~ Verify inverse functions
~ Find the Inverse of a Function
~ Determine if a function has an inverse
~ Graph a Function and its Inverses

➢ Domain of Functions

- **Domain:** ____________________

Practice: For each of the following functions, find the domain.

7. $f(x) = 6x^4 - 3x^2 + 2x - 41$

8. $g(x) = \dfrac{5}{x+4}$

9. $m(x) = \dfrac{3x}{2x^2 - 6x}$

10. $h(x) = \dfrac{5-2x}{3x^2 - 19x + 6}$

11. $l(x) = \sqrt{3-x}$

12. $k(t) = \sqrt{2x-16}$

Algebraic Operations with Functions

Four algebraic operations that we do with polynomial functions are:

Practice: Perform the indicated operations for the following functions.

$f(x) = x - 3,$ $g(x) = 2x^2 + 3x - 2,$ $h(x) = 2x^3 - 5x^2 + 6,$

13. f(x)+h(x)=

14. h(x)-g(x)=

15. f(x)·g(x)=

16. $\frac{g(x)}{f(x)} =$

Composition of Functions

The Composition of the function f with g is denoted by $f \circ g$ and is defined by the equation $(f \circ g)(x) = f(g(x))$. The domain of the composite function $f \circ g$ is the set of all x such that:

1. x is in the domain of g, and
2. g(x) is in the domain of f

Practice: *Perform the indicated operations for the following functions.*

$f(x) = x + 4 \qquad g(x) = x^2 - 3x - 4 \qquad k(x) = 4x^2 - 6x - 4 \qquad h(x) = 2x + 1$

17. Compose $f(x) \circ g(x)$

18. Compose $g(f(x))$

19. Work out $(h \circ h)(-3)$

Inverses

Let f(x) be a function. The inverse function of f(x) denoted by $f^{-1}(x)$ is a function such that

$$\boxed{f\left(f^{-1}(x)\right)=x=f^{-1}(f(x))}$$

The domain of $f(x)$ is equal to the range of $f^{-1}(x)$ and vice-versa

If (a,b) is a point in the graph of $f(x)$, then (b,a) is a point in the graph of $f^{-1}(x)$

Example: Determine if $f(x)=2x+6$ and $g(x)=\frac{x}{2}-3$ are inverse functions of each other.

Solution: In order for two functions to be inverses of each other

$$f(g(x))=x \text{ and also } g(f(x))=x$$

So, $f(g(x))=2\left(\frac{x}{2}-3\right)+6=\frac{2x}{2}-6+6=x$

and $g(f(x))=\frac{2x+6}{2}-3=\frac{\cancel{2}(x+3)}{\cancel{2}}-3=x-3+3=x$

Therefore, f and g are inverses of each other.

Practice: Determine if the following are inverses of each other

20. $f(x)=3x-7$ $\qquad g(x)=\frac{x+7}{3}$

21. $f(x) = \dfrac{2}{x-5}$ $g(x) = \dfrac{2}{x} + 5$

- **Finding Inverse Functions**

Steps To find Inverse Functions
1. Replace $f(x)$ ***with y in the equation for*** $f(x)$
2. Interchange x and y
3. Solve for y. If you get a function then f has an inverse.
4. Replace y by $f^{-1}(x)$
5. Verify by $f(f^{-1}(x)) = x = f^{-1}(f(x))$

Example: Find the inverse of $f(x) = 3x+1$

$Solution: f(x) = 3x+1$

$y = 3x+1$ $step 1$

$x = 3y+1$ $step 2$

$x-1 = 3y \Rightarrow \dfrac{x-1}{3} = y$ $step 3$

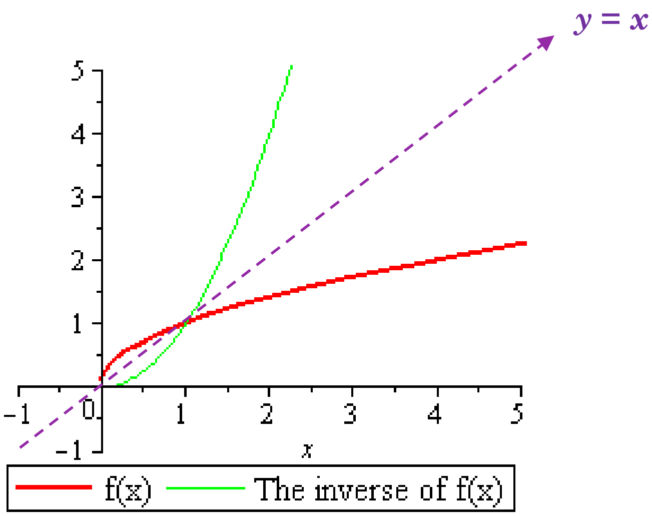
$f^{-1}(x) = \dfrac{x-1}{3}$ $step 4$

$Verify!$

$f(f^{-1}(x)) = 3\dfrac{x-1}{3} + 1 = x - 1 + 1 = x$

$f^{-1}(f(x)) = \dfrac{(3x+1)-1}{3} = \dfrac{3x}{3} = x$

The graph of an inverse $f^{-1}(x)$ function is a reflection of $f(x)$ with respect to the line $y = x$



Practice: Find the inverse of the following functions.

22. $f(x) = x^3 - 1$

23. $f(x) = \frac{4}{x} + 9$

24. $f(x)=\dfrac{2x-3}{x+1}$

25. $f(x)=\sqrt[3]{x-1}$

• Existence of Inverse Functions

There are functions that do not have inverses: Example: $f(x) = x^2, \ (-\infty, \infty)$

There are two ways we can use to decide if a function has an inverse

- **Algebraically**: If a function is 1-1 then a function has an inverse

  A function is 1-1 if and only if

$$\boxed{f(x_1) = f(x_2) \Rightarrow x_1 = x_2}$$

Example:

- $f(x) = 3x + 12 \quad is \quad 1-1 \quad since$

  if

  $f(x_1) = f(x_2) \Rightarrow 3x_1 + 12 = 3x_2 + 12 \Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2$

- $f(x) = x^2 \ \ is \ \ not \ \ 1-1 \ \ since$

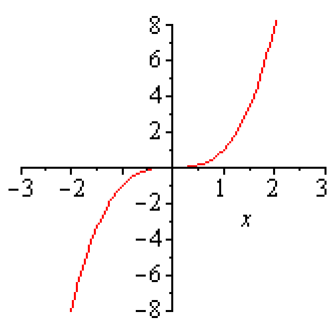
  if

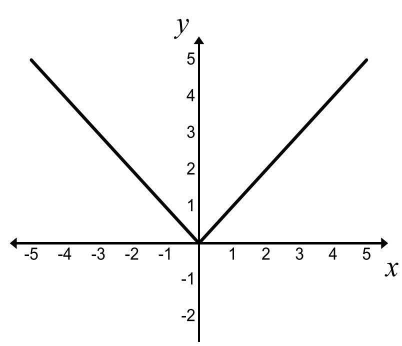
  $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \cancel{\Rightarrow} x_1 = x_2 \ since \ (2)^2 = (-2)^2 \ but \ 2 \neq -2$

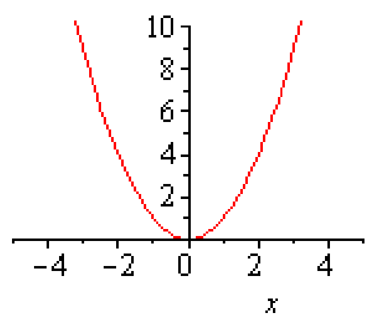
- **Graphically**: If a function passes the Horizontal Line Test then a function has an inverse.

The horizontal Line Test: A function has an inverse that is a function if there is no horizontal line that intersects the graph of f at more than one point.

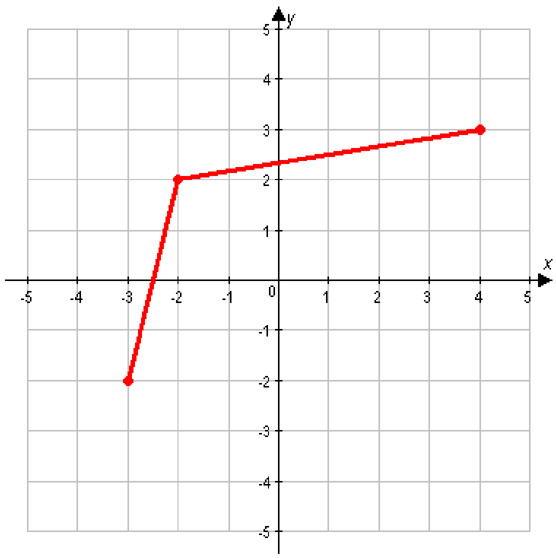
Practice: Determine if the following functions have an inverse



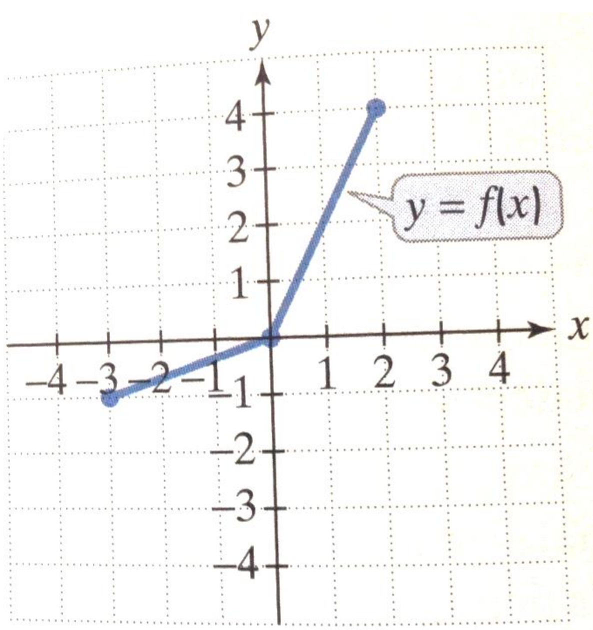




26. Use the given graph of f to draw the graph of f^{-1}



27.



Class 7: Distance and Midpoint Formulas; Circles

Objectives:

- ~ Find the Distance between two points.
- ~ Find the midpoint of a line segment
- ~ Write the standard form of a circle's equation
- ~ Give the center and radius of a circle whose equation is in standard form
- ~ Convert the general form of a circle's equation to standard form

➢ Distance & Midpoint

➢ The Distance Formula

The distance, d, between the points (x_1, y_1) and (x_2, y_2) in the rectangular coordinate system is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Then midpoint Formula

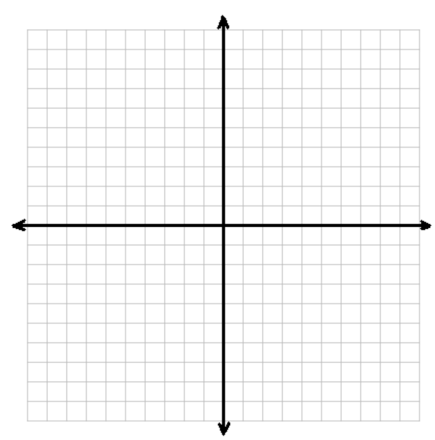
Consider a line segment whose endpoints are (x_1, y_1) and (x_2, y_2).

The coordinates of the segment's midpoint are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

1. Plot the points A(4, 6), B(-3, 2), and C (1,-5) on a coordinate system and connect them in order to find a triangle.

   a) Calculate the lengths of the three sides of the triangle.



b) Verify that this triangle is a right triangle.

c) Calculate the midpoints of the sides AB and BC. Determine the length of the segment connecting these two midpoints.

d) Is the length of side AC twice the length of the segment in part (c)?

Circles

- **A Circle is**____
- **Radius is** ____
- **The Standard Equation of a circle is** ____________________________________
- **The General Form of the Equation of a Circle is** __________________________

Practice: Write the standard equation for the circle in each of the following cases;

2. Center (0,0), $r = 8$

3. Center (-3, 5), $r = 3$

4. $Center\ (-5,-3),\ r=\sqrt{5}$

Practice: Give the center and radius of the circle described by the following equations:

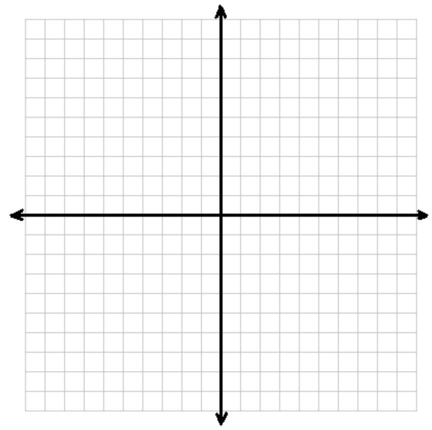
5. $(x-2)^2+(y-3)^2=16$

6. $(x+5)^2+(y-4)^2=6$

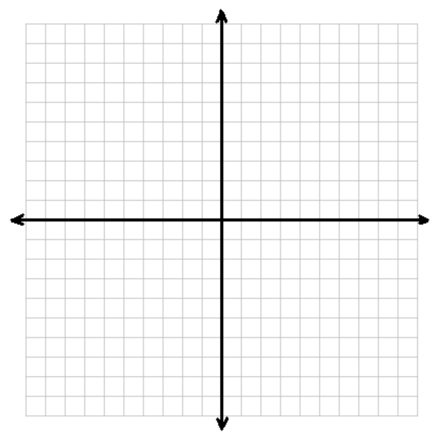
7. $(x+7)^2+y^2=13$

Practice: For each of the following, complete the square and write the equation in standard form. Then graph each equation and use it to identify the domain and range.

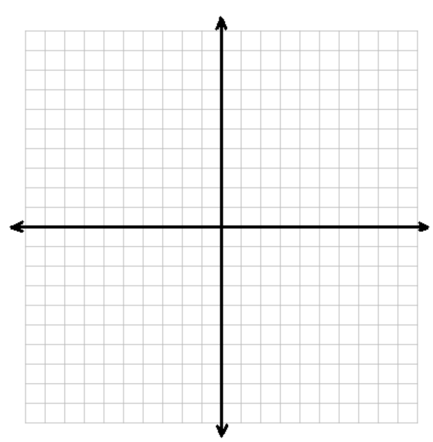
8. $x^2 + y^2 + 8x + 4y + 16 = 0$



9. $x^2 + y^2 - 6y - 7 = 0$



10. $x^2 + y^2 + 3x + 5y + \frac{9}{4} = 0$



Test 1: Summary/Questions

Class 9: Section 5.1 - Angles and Their Measurements

Objectives:

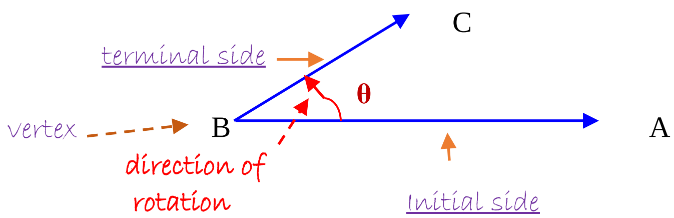
- ~ Define and draw angles
- ~ Convert angles from Degrees to Radians
- ~ Convert angles form Radians to Degrees
- ~ Use Right Triangles to Evaluate Trigonometric Functions

❖ Definitions:

- **Line**:  **Line Segment**: 
- **Ray** : one end is fixed 

  Example: The hour hand of a clock suggests a ray.

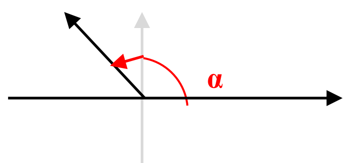
- **Angle**: is formed by two rays that have a common point.



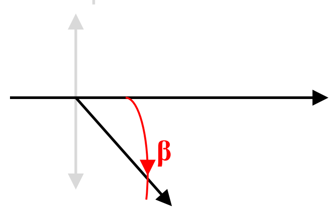
We are going to use Greek letters to name angles

- **Standard Position**: an angle is in standard position if
  - Its vertex is at the origin of a Rectangular Coordinate system
  - Its initial side lies along the positive axis.

- **Positive Angles** are angles generated by Counter- clockwise rotation



- **Negative Angles** are angles that are generated by Clock –wise rotation.



- **Quadrantal Angles** are angles whose terminal side lies on the x-axis or the y-axis.

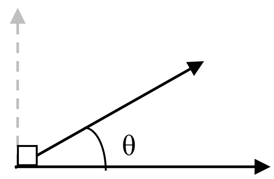
Measuring Angles

There are two ways of measuring angles: Degrees or Radians

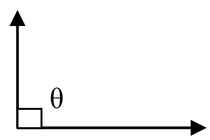
- **By Degrees (°)**
  - **One Revolution** is the rotation of a ray back to itself = **360°.**

We can classify angles by degrees:

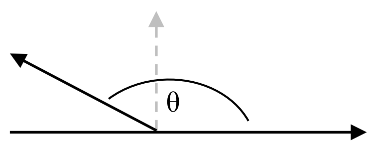
- Acute angle is angle whose measure < 90°



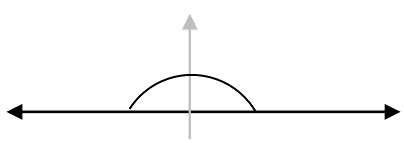
- Right angle is an angle whose measure = 90° (A Quarter of a Revolution)



- Obtuse Angle is an angle whose measure is between 90° and 180°.



- Straight angle is an angle whose measure = 180° (Half of a Rotation)

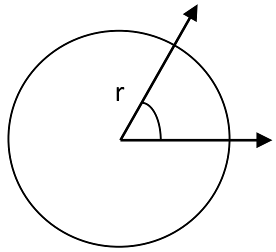


Practice: Classify the following angles:

1. 125° -
2. 160° -
3. 65° -
4. 90° -
5. 45° -
6. 180° -

- **By Radians** http://www.geogebratube.org/student/m59882

- **Central Angle**: is an angle whose vertex is at the center of a circle.

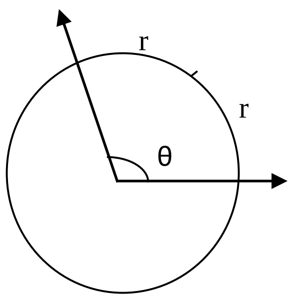


- **One Radian** is the measure of the central angle that intercepts an arc equal in length to the radius of the circle.

- **Radian Measure** is the length of the intercepted arc divided by the circle's radius

$$\boxed{\theta = \frac{Lenght\ of\ the\ intercepted\ arc}{radius} = \frac{s}{r}\ radians}$$

Example:



$$\Rightarrow \quad \theta = \frac{2r}{r} = 2\,radians$$

Practice:

7. Find the measure of the angle θ that intercepts an arc of length 15 inches in a circle of radius 6 in.

For the following find the missing quantity:

8. $r = 10\,m, \quad \theta = \frac{1}{2}\, radians, \quad s =$

9. $\theta = \frac{1}{3}\, radians, \quad s = 2\ ft, \quad r =$

10. $r = 5m, \quad s = 3m, \quad \theta =$

11. $r = 3m, \quad \theta = 120^{\circ}, \quad s =$

Relationship between Degrees & Radians

$$One\ Rotation = 360^\circ = \frac{2\pi r}{r} = 2\pi \Rightarrow \begin{cases} 360^\circ = 2\pi \\ 180^\circ = \pi \end{cases} \Rightarrow \boxed{180^\circ = \pi\ radians}$$

Conversions:
▪ **To convert degrees to radians, multiply both sides by** $\frac{\pi\ radians}{180^\circ}$
▪ **To convert from radians to degrees, multiply both sides by** $\frac{180^\circ}{\pi\ radians}$

Practice: Convert from radians to degree or degrees to radians as necessary

12. 30° =

13. 90° =

14. $\frac{\pi}{3} radians =$

15. $-\frac{5\pi}{3} radians =$

16. 1 *radian* =

17. 150° =

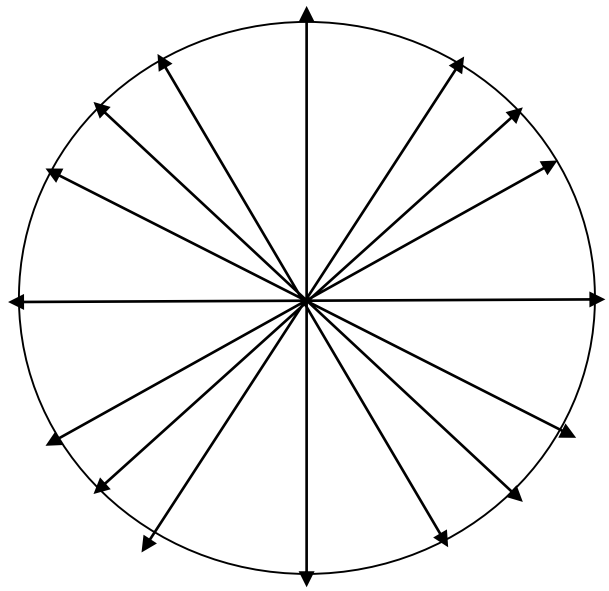
18. -45° =

19. 125° =

20. 60° =

21. $\frac{\pi}{6} radians =$

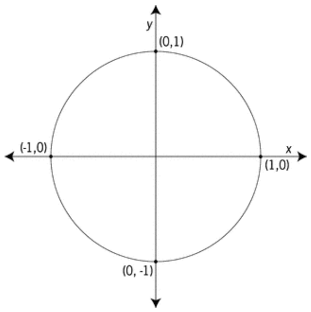
Fill the circle with the degree and radian measure

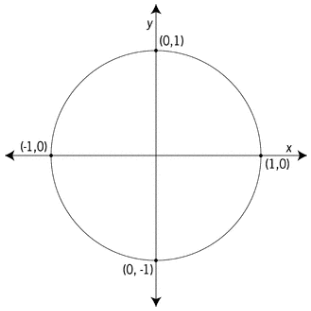


Practice: State the quadrant each angle is and then draw the angle in standard position.

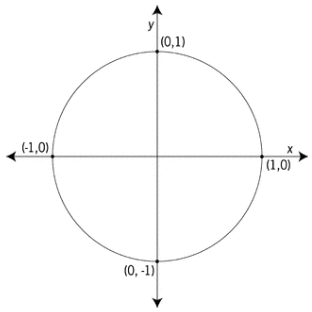
22. $\dfrac{3\pi}{2}$

23. $-\dfrac{3\pi}{4}$

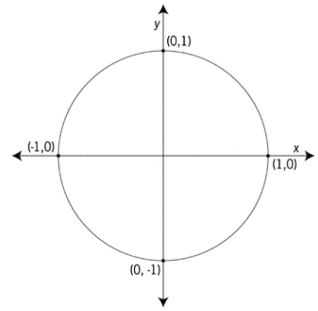




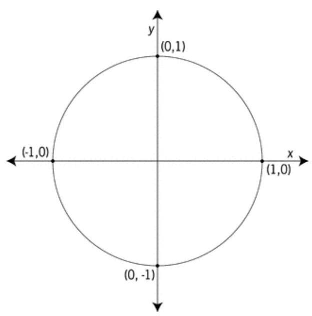
24. $\dfrac{7\pi}{3}$



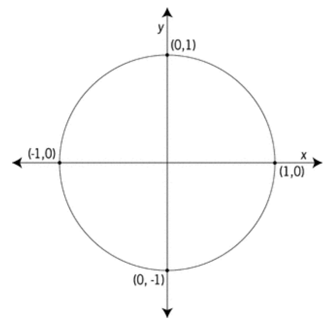
25. $\dfrac{6\pi}{4}$



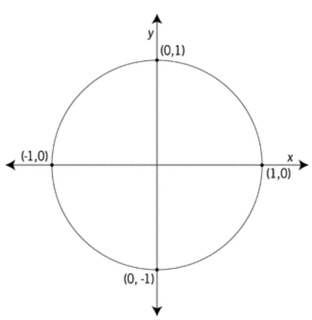
26. $-\dfrac{\pi}{4}$



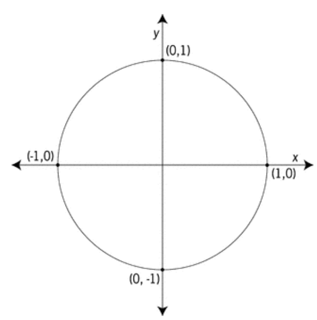
27. $\dfrac{3\pi}{4}$



28. $-\dfrac{7\pi}{3}$



29. $\dfrac{8\pi}{6}$



Trigonometric Functions

The word trigonometry comes from the greek word "trigono-metria", meaning measures of triangles.

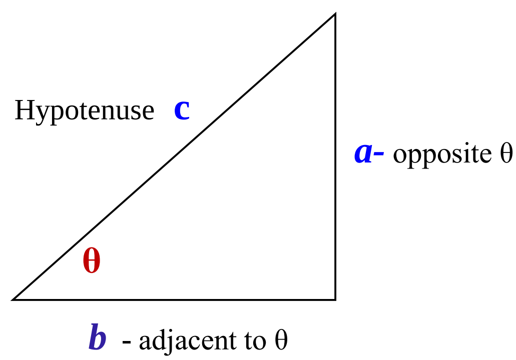
It was used in Navigation in Building Engineering and today we use it in studying the structure of DNA.

- **Trigonometric Functions.**

The inputs of these functions are measures of acute angles in right triangles and the outputs are ratios of lengths of the sides of right triangles.

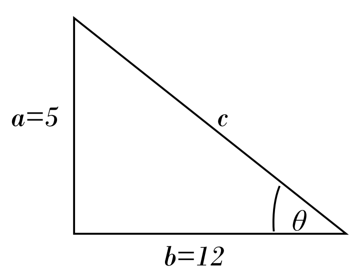
Consider the right triangle ABC with sides a, b and c.

NAME	RATIO	FUNCTION
Sine	Opposite / hypotenuse	$\sin\theta = \frac{a}{c}$
Cosine	Adjacent / hypotenuse	$\cos\theta = \frac{b}{c}$
Tangent	Opposite / adjacent	$\tan\theta = \frac{a}{b}$
Cosecant	Hypotenuse / opposite	$\csc\theta = \frac{c}{a}$
Secant	Hypotenuse / adjacent	$\sec\theta = \frac{c}{b}$
Cotangent	Adjacent / opposite	$\cot\theta = \frac{b}{a}$



The trigonometric function depends only on the size of the angle θ and not in the size of the triangle.
http://www.univie.ac.at/moe/galerie/wfun/wfun.html

Example 1: Find the value of each of the six trigonometric functions of θ for the following triangle.



Solution:

In order to find the six trigonometric functions
we need to know the lenghts of all sides of the triangle
In this case we know the lengths of two of the sides,
but we can find the length of the other side by usng the
Pythagorean Theorem: $c^2 = a^2 + b^2$

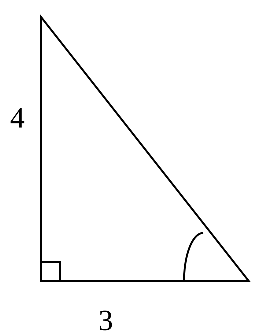
So $c^2 = 5^2 + 12^2 \Rightarrow c^2 = 169 \Rightarrow c = \sqrt{169} \Rightarrow c = 13$

Hence:

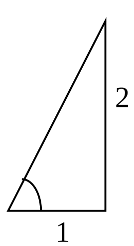
$$\left.\begin{array}{lll} \sin\theta = \dfrac{opp}{hyp} = \dfrac{5}{13} & \rightarrow & \csc\theta = \dfrac{hyp}{opp} = \dfrac{13}{5} \\ \cos\theta = \dfrac{adj}{hyp} = \dfrac{12}{13} & \rightarrow & \sec\theta = \dfrac{hyp}{adj} = \dfrac{13}{12} \\ \tan\theta = \dfrac{opp}{adj} = \dfrac{5}{12} & \rightarrow & \cot\theta = \dfrac{adj}{opp} = \dfrac{12}{5} \end{array}\right\} \textit{These three are the reciprocal identities}$$

Practice:

30. Find the value of each of the six trigonometric functions of θ for the following triangle.

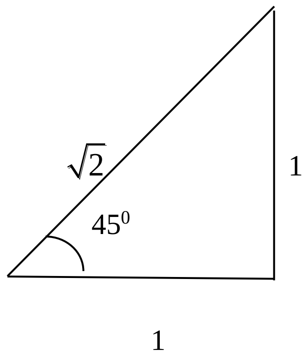


32. Find the value of each of the six trigonometric functions of θ for the following triangle.



- **Special Angles** 30°, 45°, 60° **or** $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$

45° *or* $\frac{\pi}{4}$ is constructed by constructing an isosceles right triangle.



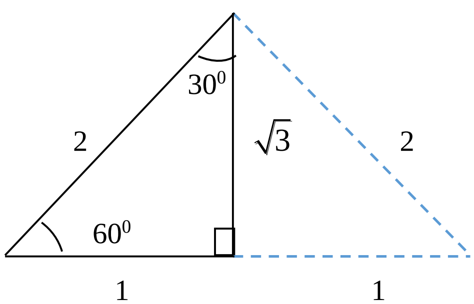
$\sin\frac{\pi}{4} =$

$\cos\frac{\pi}{4} =$

$\tan\frac{\pi}{4} =$

- 30° *or* $\frac{\pi}{6}$ *and* 60° *or* $\frac{\pi}{3}$ is constructed by constructing an equilateral triangle.

Example:



$\sin\frac{\pi}{3} =$ $\sin\frac{\pi}{6} =$

$\cos\frac{\pi}{3} =$ $\cos\frac{\pi}{6} =$

$\tan\frac{\pi}{3} =$ $\tan\frac{\pi}{6} =$

Special Angles/ Special Identities

θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$
sin θ			
cos θ			
tan θ			

Reciprocal Identities:

$\sin\theta = \frac{1}{\csc\theta}$ $\csc\theta = \frac{1}{\sin\theta}$

$\cos\theta = \frac{1}{\sec\theta}$ $\sec\theta = \frac{1}{\cos\theta}$

$\tan\theta = \frac{1}{\cot\theta}$ $\cot\theta = \frac{1}{\tan\theta}$

- ***Pythagorean Identities:***

$\boxed{\sin^2\theta + \cos^2\theta = 1}$ $\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$

$\boxed{1 + \tan^2\theta = \sec^2\theta}$

$\boxed{1 + \cot^2\theta = \csc^2\theta}$ $\boxed{\cot\theta = \frac{\cos\theta}{\sin\theta}}$

Example: Given that $\sin\theta = \frac{1}{2}$ and θ is acute, find $\cos\theta$

Solution: *By the Pythagorean Identitity*:

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \left(\frac{1}{2}\right)^2 + \cos^2\theta = 1 \Rightarrow \left(\frac{1}{4}\right) + \cos^2\theta = 1 \Rightarrow \cos^2\theta = \frac{3}{4} \Rightarrow \cos\theta = \sqrt{\frac{3}{4}} \Rightarrow \boxed{\cos\theta = \frac{\sqrt{3}}{2}}$$

Practice: Use identities to find the trigonometric function.

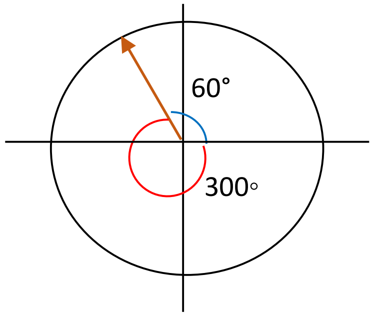
33. Find $\sin\theta$ if $\cos\theta = \frac{7}{8}$

34. Find $\tan\theta$ if $\sin\theta = \frac{6}{7}$

Co-terminal Angles

Coterminal angles are two angles who have the same and terminal side but possible different rotation

Example:



Every angle has infinitely many co-terminal angles

$$\boxed{\theta \pm k(2\pi)\ \ or\ \ \theta \pm 360^{\circ} k}$$

Practice: Find a positive angle less than 2π that is co-terminal with each of the following.

30. 400°

31. -135°

Class 10: Trigonometric Functions of any Angle

Objectives:

- Trigonometric functions of any angle/Definition
- Use the signs of the trigonometric functions
- Reference Angle
- Applications of Trigonometric Functions

➢ Trigonometric Functions of Any Angle

In nature there are many phenomena that occur in repetition, such as the tides, number of hours of daylight etc. In the remainder of the semester we are going to see how trigonometric functions are used to model those kind of phenomena and for that reason, we need to move beyond right triangles.

Definition of Trigonometric Functions of any Angle:

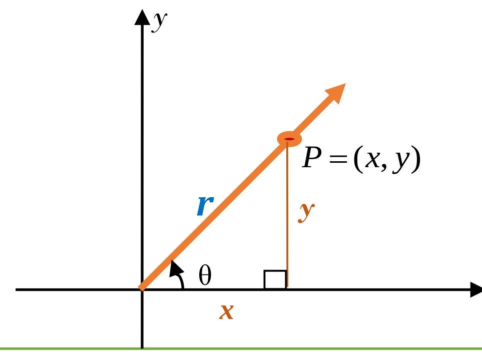
Let θ be any angle in standard position and let $P=(x,y)$ be a point on the terminal side of θ. If $r=\sqrt{x^2+y^2}$ is the distance from (0,0) to (*x*, *y*). The six trigonometric functions of θ are defined by the following ratios.

Because $P=(x,y)\neq(0,0)\ \Rightarrow\ r\neq 0$

$$\sin\theta=\frac{y}{r} \qquad\qquad \csc\theta=\frac{r}{y} \quad y\neq 0$$

$$\cos\theta=\frac{x}{r} \qquad\qquad \sec\theta=\frac{r}{x} \quad x\neq 0$$

$$\tan\theta=\frac{y}{x} \quad x\neq 0 \qquad \cot\theta=\frac{x}{y} \quad y\neq 0$$



Example: Let $P=(-3,-4)$ be a point in the terminal side of θ. Find the value of the six trig. functions

In order to evaluate the six trigonometric functionswe need to know r, x and y.

In our case, x and y are known so by uing the Pythagorean Theorem we can find r.

$$Hence:\ r^2=\sqrt{x^2+y^2}=\sqrt{(-3)^2+(-4)^2}=\sqrt{25}=5$$

$$\sin\theta=\frac{y}{r}=\frac{-4}{5} \qquad \csc\theta=\frac{r}{y}=\frac{5}{-4}$$

$$\cos\theta=\frac{x}{r}=\frac{-3}{5} \qquad \cos\theta=\frac{r}{x}=\frac{5}{-3}$$

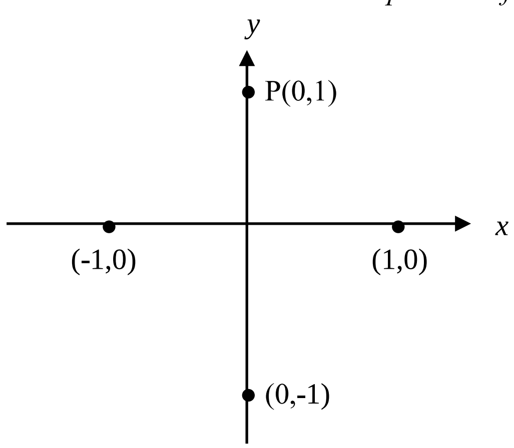
$$\tan\theta=\frac{y}{x}=\frac{-4}{-3}=\frac{4}{3} \qquad \cot\theta=\frac{x}{y}=\frac{-3}{-4}=\frac{3}{4}$$

Practice:

34. Let $P=(1,-3)$ be a point in the terminal side of θ. Find the value of the six trigonometric functions.

35. Let $P=(-12,5)$ be a point in the terminal side of θ. Find the value of the six trigonometric functions.

Quadrantal Angles: Lets find the values of trigonometric functions for the quadrantal angles.
To find the values of the trigonometric functions of quadrantal angles we pick a point on the axis (x or y depending on the angle we want). Since the values of the trigonometric functions do not depend on the value of θ and not in the distance the point lies from (0,0) we can chose a point that is 1 unit from the origin.

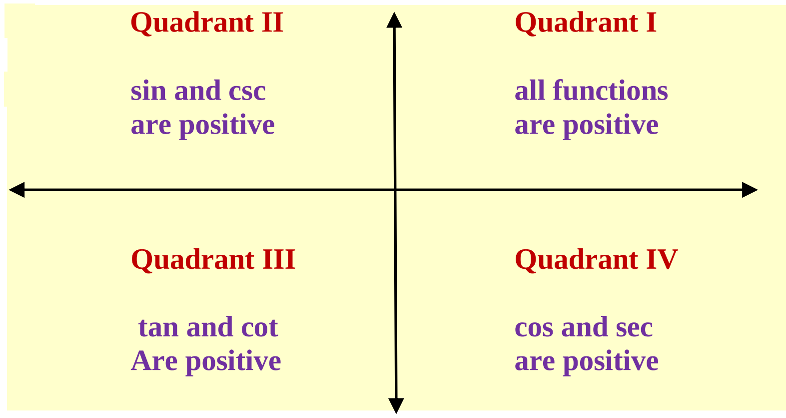


θ	**0** **2π**	**90°** **π/2**	**180°** **π**	**270°** **3π/2**
sinθ				
cosθ				
tanθ				

The Signs of the Trigonometric Functions

If θ is not a quadrantal angle then the signs of the values of the trigonometric functions depend on the quadrant the angle lies.

The table summarizes the signs of the trigonometric functions



Here is an easy way to remember:

I	II	III	IV
All	**S**tudents	**T**ake	**C**alculus
All	sin/csc	tan/cot	cos/sec

3 . Given $\tan\theta = -\frac{1}{3}$ and $\cos\theta < 0$, find $sin\theta$ and $sec\theta$.

4. Given $\csc\theta = -4$ and $\tan\theta > 0$, find the exact values of the other trigonometric functions.

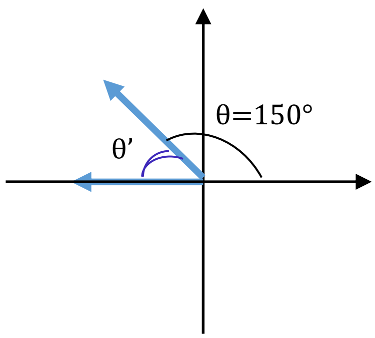
Reference Angles

http://math.cowpi.com/geogebra/reference_angle.html

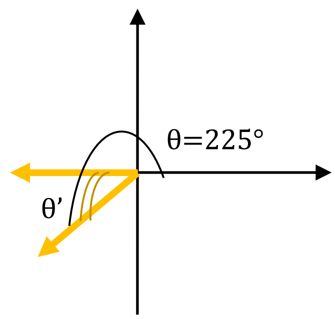
We will often evaluate trigonometric functions of positive angles greater than 90° and all negative angles by making use of a positive acute angle. This positive acute angle is called Reference Angle.

Definition of a Reference Angle
Let θ be a non-acute angle in standard position that lies in a quadrant.
Its **reference angle** is the positive acute angle θ' formed by the terminal side of θ and the x-axis.

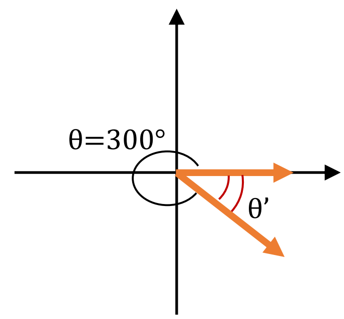
Example:



θ'=180°-150°=30°



θ'=225°-180°=45°



θ'=360°-300°=60°

So for an angle θ< 360° :

If	$90^\circ < \theta < 180^\circ$	*If*	$180^\circ < \theta < 270^\circ$	*If*	$270^\circ < \theta < 360^\circ$
then	$\theta' = 180^\circ - \theta$	*then*	$\theta' = \theta - 180^\circ$	*then*	$\theta' = 360^\circ - \theta$

Example: Find the reference angle of θ = 210°

Practice: Find the reference angles of the following angles:

5. $\theta = \dfrac{7\pi}{4} \rightarrow \theta' =$

6. $\theta = -240^\circ \rightarrow \theta' =$

7. $\theta = 3.6 \rightarrow \theta' =$

Finding reference angles for angles greater than 360° (2π)

1. Find a positive angle α greater than 360° or 2π that is coterminal with the given angle.
2. Draw α in standard position.
3. Use the drawing to find the reference angle.

Example: Find the reference angle of θ = 665°

$$\text{Solution: } 665^\circ - 360^\circ = 305^\circ, \quad 305^\circ \textit{ lies in the IV quadrant}$$
$$\theta' = 360^\circ - 305^\circ \Rightarrow$$
$$\theta' = 55^\circ$$

Practice: Find the reference angles for the following angles

8. $\theta = \dfrac{15\pi}{4} \rightarrow \theta' =$

9. $\theta = -\dfrac{11\pi}{3} \rightarrow \theta' =$

θ	0° 2π	30° $\frac{\pi}{6}$	45° $\frac{\pi}{4}$	60° $\frac{\pi}{3}$	90° $\frac{\pi}{2}$	180° π	270° $\frac{3\pi}{2}$
sin θ							
cos θ							
tan θ							

Now we are ready to find the value of any angle by using reference angles.

The values of the trigonometric functions of a given angle θ are the same as the values of the trigonometric functions of the reference angle θ' except possibly for the sign.
http://www.ronblond.com/MathGlossary/Division04/TrigCircle/

Evaluating Trigonometric Functions using Reference Angles
1. *Find the associated reference angle, θ' and the function value for θ'.*   2. *Use the quadrant in which θ lies to find the appropriate sign to the function in step 1.*

Example: Find the exact value of $\cos\frac{4\pi}{3}$

Solution:

$$\frac{4\pi}{3} \textit{ lies on the third quadrant, therefore } \cos\theta \textit{ would be negative.}$$

$$\textit{The reference angle of } \frac{4\pi}{3} \textit{ is: } \theta' = \frac{4\pi}{3} - \pi = \frac{4\pi}{3} - \frac{3\pi}{3} = \frac{4\pi - 3\pi}{3} = \frac{\pi}{3}$$

$$\textit{Hence, } \cos\frac{4\pi}{3} = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

Practice: Use identities to find the trigonometric function.

10. Find the exact value of $\tan(-210^\circ)$

11. Find the exact value of $\csc\frac{11\pi}{4}$

For each of the following find the values of the other trigonometric functions and find θ.

12. $\csc\theta = -\dfrac{5}{2}$ *and* $\cos\theta < 0$

13. $\cot\theta = -1.4705$ *and* $\cos\theta > 0$

Inverse Trigonometric Functions

Inverse trigonometric functions have as in input a ratio, or decimal and as an output an angle.

Example: When evaluating the inverse trig function, $\sin\theta = .2345$, we find the angle θ whose sin is equal to .2345

Practice

14. Find the exact value of $\arcsin\frac{\sqrt{3}}{2}$

15. Find the exact value of $\cos^{-1}\frac{\sqrt{2}}{2}$

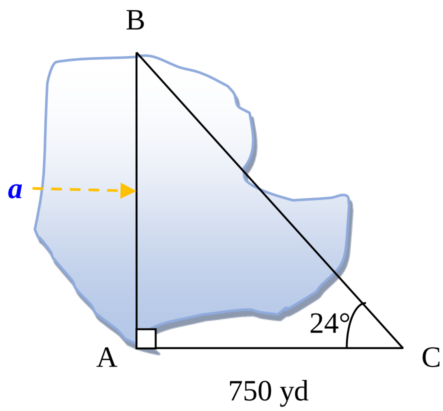
16. Find the exact value of $\sin^{-1}3$

17. Find the exact value of $\arccos\left(-\frac{1}{2}\right)$

18. Find the angle θ, if $\sin\theta = .2845$

19. Find the angle θ, if $\cos\theta = -.5845$

20. The distance *a* across a lake is unknown. To find this distance, a surveyor took the measurements shown. What is the distance across the lake?



21. A flagpole that is 14 m tall casts a shadow 10 m long. Find the angle of elevation of the sun to the nearest degree.

22. A wire is 13.8 yards long and is attached from the ground to a pole 6.7 yards above ground. Find the angle, to the nearest tenth of a degree that the wire makes with the ground.

Class 11: Trig Functions of Real Numbers & their Graphs

Objectives:

- Trigonometric functions of real numbers
- Recognize Domain and Range of Sin and Cos functions
- Use of Even and Odd trigonometric Functions
- Use of Periodic Properties
- Graph the sine and cosine functions and their transformations

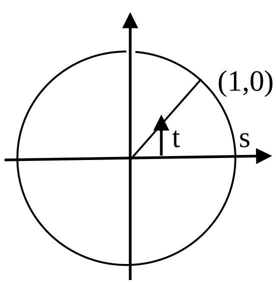
Trigonometric Functions of Real Numbers

Cycles govern many aspects of our lives such as sleep patterns, seasons, tides etc. All follow regular, predictable cycles.

In this section we are going to see why trigonometric functions are used to modes such phenomena.

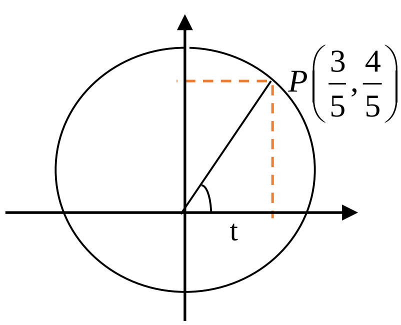
Until now we have considered trigonometric functions of angles. To define trigonometric functions of real numbers rather than angles we use a unit circle.

Unit Circle: is a circle of radius 1 centered at (0,0) whose equation is $x^2 + y^2 = 1$



The central angle t is measured in radians. The arc length $s = rt = 1t = t$

Therefore the radian measure of the central angle is equal to the length of the intercepted arc



Example: P corresponds to a real number *t*.

So $\sin t = \dfrac{\frac{4}{5}}{1} = \dfrac{4}{5}$ and $\cos t = \dfrac{\frac{3}{5}}{1} = \dfrac{3}{5}$

Because the sine function is the y-coordinate and the cosine is the x-coordinate the values of the trigonometric functions at the real number *t* are $\sin t = \dfrac{4}{5}$ *and* $\cos t = \dfrac{3}{5}$

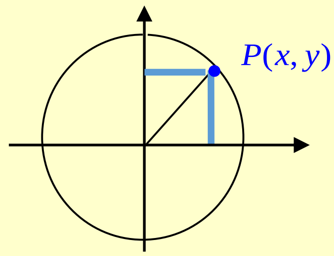
Definitions of the Trigonometric Functions in Terms of a Unit Circle

If t is a real number and $P=(x, y)$ is a point on the unit circle that corresponds to t, then

$$\sin t = y \qquad \csc\theta = \frac{1}{y} \quad y \neq 0$$

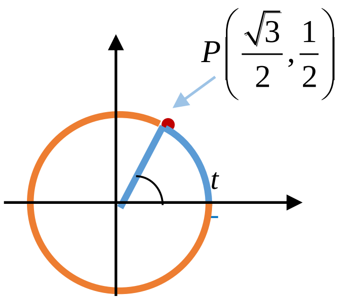
$$\cos t = x \qquad \sec\theta = \frac{1}{x} \quad x \neq 0$$

$$\tan t = \frac{y}{x} \quad x \neq 0 \qquad \cot\theta = \frac{x}{y} \quad y \neq 0$$

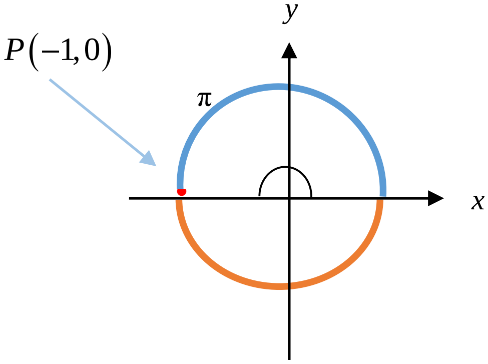


Practice:

1. Use the figure below to find the values of the trigonometric functions:



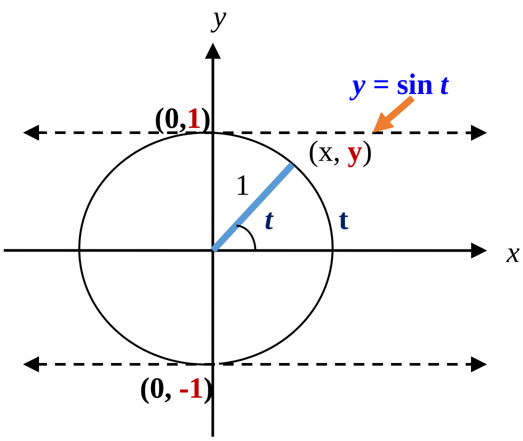
2. Use the figure below to find the values of the trigonometric functions:



The value of a trigonometric function at a real number *t* is its value at an angle of *t* radians

http://www.analyzemath.com/unitcircle/unit_circle_applet.html

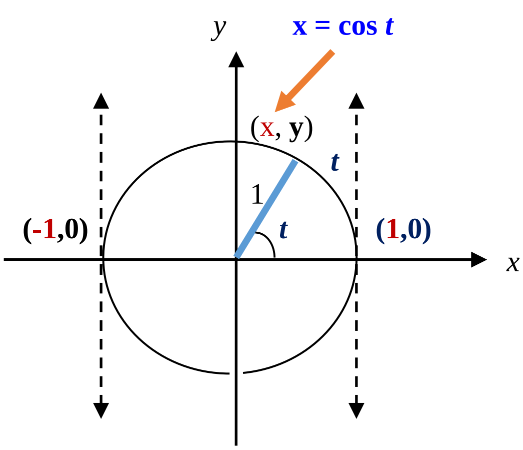
❖ $\boxed{y = \sin t}$



Domain:

Range:

❖ $\boxed{y = \cos t}$



Domain:

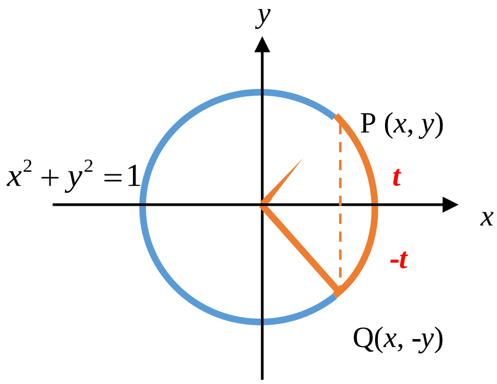
Range:

- **Domain** of sine and cosine functions is the set of all real numbers $(-\infty, \infty)$
- **Range** of sine and cosine functions is the set of real numbers from -1 to 1, $[-1, 1]$

Even and Odd Trigonometric Functions

- **Even Function**: $f(t) = f(-t)$ $\quad Ex: f(x) = x^2$ since $f(-x) = (-x)^2 = x^2 = f(x)$
- **Odd Function**: $f(t) = -f(t)$ $\quad Ex: f(x) = x^3$ since $f(-x) = (-x)^3 = -x^3 = -f(x)$

Now let's look at the sine and the cosine functions



P has coordinates: $(\cos t,\ \sin t)$

Q has coordinates $(\cos(-t),\ \sin(-t))$

But as we can see from the picture the x coordinate is the same for both points:

Therefore, $\cos(-t) = \cos t \;\Rightarrow$ Cosine function is an even function

Also $\sin(-t) = -\sin t \;\Rightarrow$ Sine function is an odd function

Even and Odd Trigonometric Functions

The cosine and secant functions are even functions

$\cos(-t) = \cos t$ $\qquad \sec(-t) = \sec t$

The sine, sosecant, tangent and cotangent functions are odd functions

$\sin(-t) = -\sin t$ $\qquad \csc(-t) = -\csc t$

$\tan(-t) = -\tan t$ $\qquad \cot(-t) = -\cot t$

Practice: Find the exact value of each trigonometric function:

3. $\sin(-\frac{\pi}{3}) =$

4. $\tan\left(-60^\circ\right) =$

Periodic Functions

Certain phenomena in nature repeat in cycles. For example the ocean level at a beach varies from low tide to high tide and then back to low tide again every 12 hours.

If f (t) represents the ocean level at time t, then the ocean level would be the same after 12 hours
So $f(t) = f(t + 12)$

The word periodic means that this tidal behavior repeats infinitely.
The period, 12hrs is the time it takes to complete for one cycle

Definition of the Periodic Functions

A function f is **periodic** if there exists a positive number p such that

$$f(t+p) = f(t)$$

For all t in the domain of f.
The smallest positive number **p** for which f is periodic is called the **period** of f.

The trigonometric functions are periodic since if we start at any point P and travel along the circle at a distance of 2π units, then we will return to the same point.

Periodic Properties of Trigonometric Functions

Sine and Cosine Functions have period of **2π.**

$\sin(t+2\pi) = \sin t$ or $\sin(t+2n\pi) = \sin t$

$\cos(t+2\pi) = \cos t$ $\cos(t+2n\pi) = \cos t$

Tangent and Cotangent Functions have period **π.**

$\tan(t+\pi) = \tan t$ or $\tan(t+n\pi) = \tan t$

$\cot(t+\pi) = \cot t$ $\cot(t+n\pi) = \cot t$

Practice: Find the exact values of each trigonometric function

5 . $\cos(405^\circ) =$

6. $\sin\dfrac{7\pi}{3} =$

The Graph of Sine

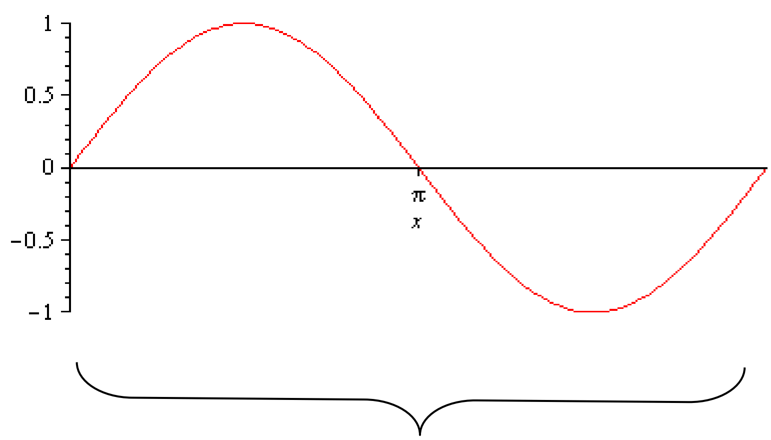
$$\boxed{y = \sin x}$$

We graph a function by plotting points that satisfy that function in the coordinate system. So we are going to graph y = sinx by listing some points on the graph.

Since the sine function is a periodic function with period 2π we are going to graph the function on [0, 2π]

x	**0**	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
y=sinx	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

<u>When we plot the points in the coordinate system we get the graph below:</u>



One period (2π)

Domain	Range	Period	Odd Function	x-intercepts	Max/Min:

General Equation of Sine Function: $\boxed{y = A\sin(Bx - C) + D}$

http://archive.geogebra.org/en/upload/files/couture_daniel/TrigFunctions.html

Example: Determine the period, phase shift, and amplitude for $y = 3\sin\left(2x - \frac{\pi}{3}\right)$ and graph the function.

Graphing Variations of y = sinx.

1. Identify the amplitude and the period
2. Find the values of x for the five key points – three intercepts, max and min. Start with the value of x where the cycle begins and add quarter periods - $\frac{period}{4}$ - to find successive values of x.
3. Find the values of y for the five key points by evaluating the function at each of the above x values.
4. Connect the five key points with a smooth curve and graph one complete cycle of the given function.
5. Extend the graph in step 4 to the left or right as desired.

Practice: Graph the function $y = -2\sin\left(4\pi x + \pi\right) + 3$

The Graph of Cosine

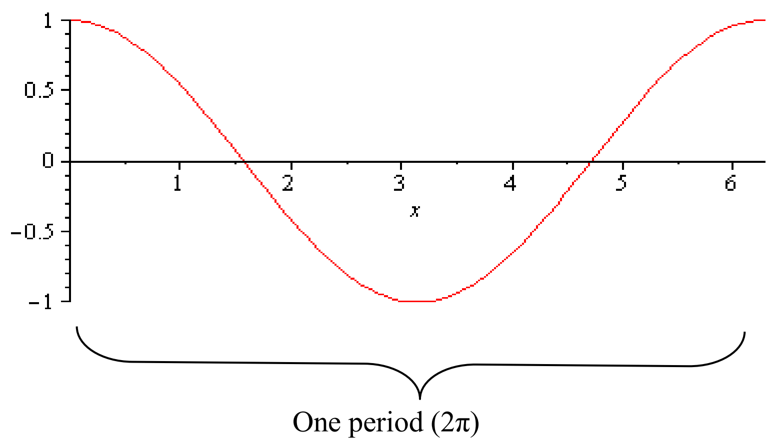
$$\boxed{y = \cos x}$$

We are going to graph y = cosx also by listing some points on the graph.

Since the cosine function is a periodic function with period 2π we are going to graph the function on [0, 2π]

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
y = cosx	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

<u>When we plot the points in the coordinate system we get the graph below:</u>



Domain	Range	Period	Odd Function	x-intercepts	Max/Min:

General Equation of Sine Function: $\boxed{y = A\cos(Bx - C) + D}$

If we pay attention to both graphs of sinx and cosx we see that the graph of y = cosx is the graph s y = sinx with a phase shift of $-\frac{\pi}{2}$.

i.e., $\boxed{\cos x = \sin\left(x + \frac{\pi}{2}\right)}$

Example: Determine the period, phase shift, and amplitude for $y = -4\cos(2x - \pi)$ and graph the function.

Practice: Graph the function $y = -2\cos(4\pi x + \pi) + 3$

Class 12: Inverse Trigonometric Functions and Applications

Objectives:

- ~ Inverse Trigonometric Functions
- ~ Solve a Right Triangle
- ~ Application Of Trigonometric Functions

➢ Inverse Trigonometric Functions

RECALL:

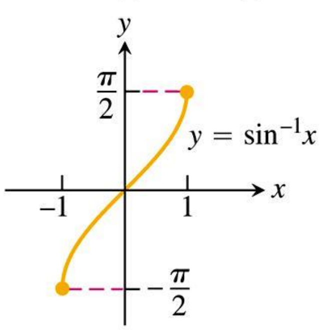
If the graph passes the horizontal line test then the function has an inverse function

If a point (*a*, *b*) is on the graph of *f*, then the point (*b*, *a*) is on the graph of f^{-1}.

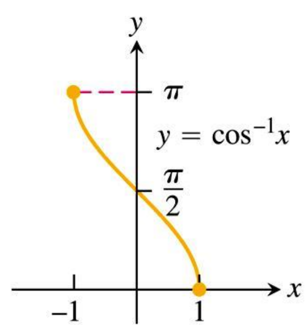
The graph of f^{-1} is a reflection of the graph of *f* about the line $y = x$.

Graphs of Inverse Trigonometric Functions:

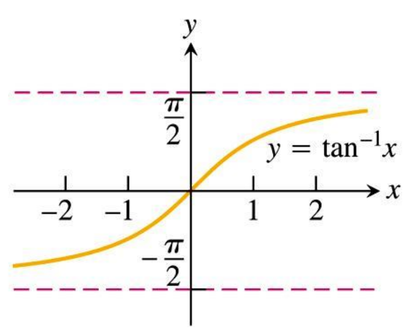
Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$



Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



❖ Finding Exact Values:

Exact values of $y = \sin^{-1} x$ can be found by thinking of $\sin^{-1} x$ as the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x.

Finding Exact Values

1. Let $\theta = \sin^{-1} x$
2. $\theta = \sin^{-1} x$ *means* $\sin\theta = x$ *where* $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
3. Use the table to find θ *in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Practice: Find the exact value of each of the following:

1. $\arcsin \frac{\sqrt{3}}{2} =$

2. $\sin^{-1} \frac{\sqrt{2}}{2} =$

3. $\sin^{-1} 3 =$

4. $\arcsin\left(-\frac{1}{2}\right) =$

5. $\arccos\left(-\frac{1}{2}\right) =$

6. $\cos^{-1} \frac{\sqrt{2}}{2} =$

7. Find the exact value of $\cos^{-1}(-1) =$

8. Find the exact value of $\arccos\left(-\frac{\sqrt{3}}{2}\right) =$

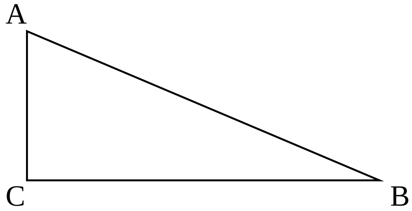
9. Find the exact value of $\arctan\left(-\frac{\sqrt{3}}{3}\right) =$

10. Find the exact value of $\tan^{-1}(0) =$

11. Find the exact value of $\tan^{-1}(-1) =$

Solving Right Triangles

Finding the missing lengths of its sides and the measures of its angles.

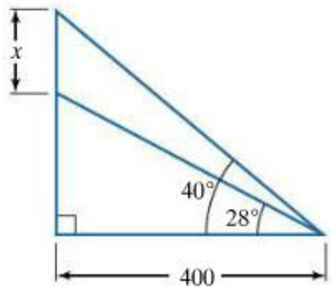


1. Let A = 62.7° and a = 8.4. Solve the right triangle shown below rounding to two decimal place

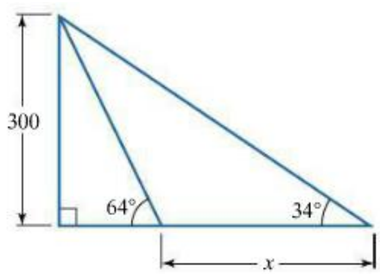
2. Let B = 23.8˚, and b = 40.5

Find x to the nearest whole unit

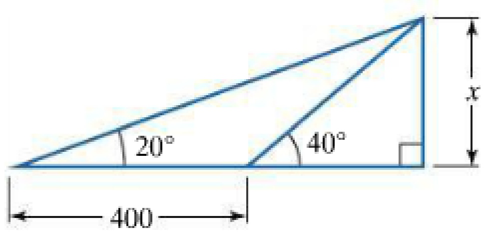
3.



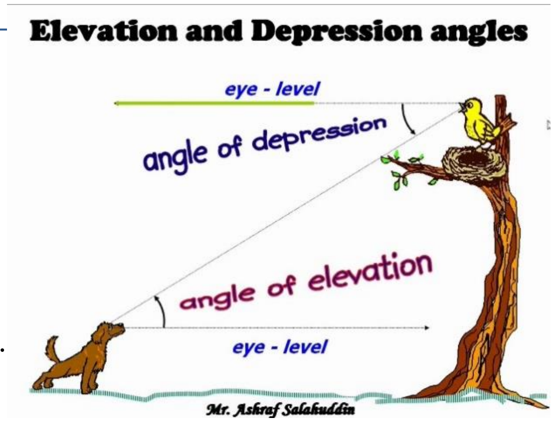
4.



5.



Angle of Elevation and Angle of Depression



6. From a point on a level ground 80 ft from the base of Eiffel Tower, the angle of elevation is 85.4°. Approximate the height of the Eiffel Tower to the nearest foot.

7. A wire is 13.8 yards long and is attached from the ground to a pole 6.7 yards above ground. Find the angle, to the nearest tenth of a degree that the wire makes with the ground.

8. You are standing on level ground 800 feet from Mt. Rushmore, looking at the sculpture of Abraham Lincoln's face. The angle of elevation to the bottom of the sculpture is 32° and the angle of elevation to the top is 35°. Find the height of the sculpture of Lincoln's face to the nearest tenth of a foot.

9. A person flying a kite holds the string 4 ft above ground level. The string of the kite is taut and makes an angle of 60° with the horizontal. Approximate the height of the kite above level ground if 500 ft of string is played out.

10. A hot air balloon rises vertically, its angle of elevation from a point P on level ground 110 km from a point Q directly underneath the balloon changes from 19.20° to 31.50°. Approximate how far does the balloon rise during this period?

11. An airplane flying at an altitude of 10,000 feet passes directly over a fixed object on the ground. One minute later, the angle of depression of the object is 42°. Approximate the speed of the airplane to the nearest mile per hour.

Class 14 – Trigonometric Identities

Objectives:

~ Use various methods to verify Trigonometric Identities

Verifying Trigonometric Identities

1. Learn Fundamental Identities (see book cover)

Be aware of equivalent forms of identities i.e., $\sin^2 x+\cos^2 x=1$ can also be $\sin^2 x=1-\cos^2 x$

2. Start with the more complicated side and simplify

DO NOT CROSS "=" sign.

3. May help to express all functions in terms of sin or cos then simplify.

4. May have to factor or use algebra

$$i.e., \frac{1}{\sin x}+\frac{1}{\cos x}=\frac{\cos x+\sin x}{\sin x\cos x} \quad \text{or} \quad \sin^2 x+2\sin x+1=(\sin x+1)^2$$

5. Keep in mind that we cannot cross sides and think about manipulating expressions

$$i.e., \frac{1+\sin x}{Denum}\left(\frac{1-\sin x}{1-\sin x}\right)=\frac{1-\sin^2 x}{Denum}=\frac{\cos^2 x}{Denum}$$

Factoring:

- $\sin^2 x-1=(\sin x-1)(\sin x+1)$
- $4\tan^2 x+\tan x-3=(4\tan x-3)(\tan x+1)$
- $\csc^2 x+\cot x-3=(1+\cot^2)+\cot x-3$

Fundamental Identities:

- **Reciprocal Id:** $\sin x=\frac{1}{\csc x}$, $\cos x=\frac{1}{\sec x}$, $\tan x=\frac{1}{\cot x}$
- **Quotient Id:** $\tan x=\frac{\sin x}{\cos x}$, $\cot x=\frac{\cos x}{\sin x}$
- **Pythagorean Id:** $\sin^2 x+\cos^2 x=1$, $\tan^2 x+1=\sec^2 x$, $\cot^2 x+1=\csc^2 x$
- **Even Odd Id:** $\sin(-x)=-\sin x$, $\cos(-x)=\cos x$, $\tan(-x)=-\tan x$
  $\csc(-x)=-\csc x$, $\sec(-x)=\sec x$, $\cot(-x)=-\cot x$

Practice

1. Verify: $\csc x \tan x = \sec x$

2. Verify: $\cos x \cot x + \sin x = \csc x$

3. Verify: $\sin x - \sin x \cos^2 x = \sin^3 x$

4. Verify: $\tan^4 x = \tan^2 x \sec^2 x - \tan^2 x$

5. $\dfrac{1+\cos x}{\sin x}=\csc x+\cot x$

6. Verify: $\dfrac{\sin x}{1+\cos x}+\dfrac{1+\cos x}{\sin x}=2\csc x$

7. Verify: $\tan x+\cot x=\sec x\csc x$

8. $$\frac{1}{1+\sin x}+\frac{1}{1-\sin x}=2+2\tan^2 x$$

9. $$\frac{\cot^2 x}{1+\csc x}=\frac{1-\sin x}{\sin x}$$

10. $$\sin^3 x\cos^4 x=(\cos^4 x-\cos^6 x)\sin x$$

Class 14 – Trigonometric Equations

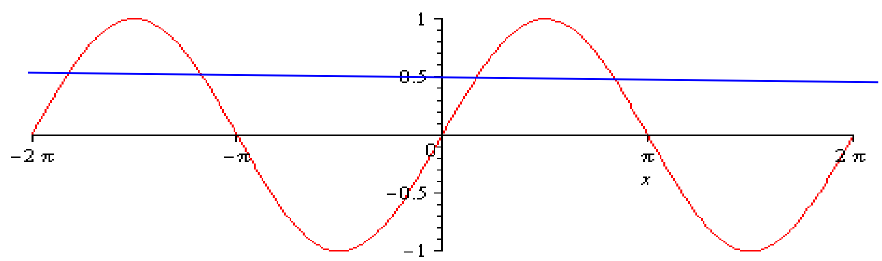
Trigonometric Equation is an equation that contains a trigonometric expression with a variable.

So far we have seen some trigonometric equations that are true for every value of the variable. Such equations we call them identities.

But there are other equations that are true only for some values of the variable. The values that satisfy such equations are the solutions of those equations.

Example: $\sin x = \frac{1}{2}$, One of the solutions is $\frac{\pi}{6}$ and another would be $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Since $\sin x$ is positive in the first and second quadrant



Because of the periodic nature of sinx there are infinitely many solutions of the form $\boxed{x = \frac{\pi}{6} + 2n\pi}$

Practice: **Solve the following equations:**

11. $5\sin x = 3\sin x + \sqrt{3}$

12. $4\cos^2 x - 3 = 0 \qquad 0 \le x \le 2\pi$

13. $2\sin^2 x - 3\sin x + 1 = 0 \qquad 0 \le x \le 2\pi$

14. $\sin x \tan x = \sin x \qquad 0 \le x \le 2\pi$

15. $2\sin^2 x - 3\cos x = 0 \qquad 0 \le x \le 2\pi$

16. $\cos^2 x + 5\cos x + 3 = 0 \qquad 0 \le x \le 2\pi$

17. $\tan 2x = \sqrt{3}$

18. $\sin \frac{x}{3} = \frac{1}{2} \qquad 0 \le x \le 2\pi$

19. $\cos 2x + \sin x = 0 \qquad 0 \le x \le 2\pi$

Class 15: The Law of Sines

Objectives:

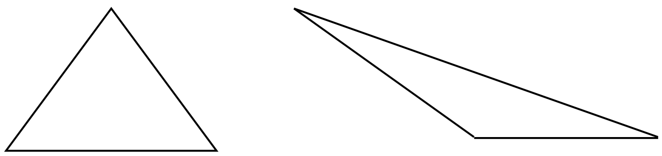
- Use the Law of Sines and Cosines to solve oblique triangles
- Solve applied problems using the Law of Sines and Cosines

Application: Two fire lookout stations are 13meters apart, with B directly east of A. Both stations spot a fire. The bearing of the fire from A is N35°E and from B is N49°W. How far is the fire from station B?

The Law of Sines

- **Oblique Triangle** is a triangle that does not contain a right angle.

Ex:



Note: In this triangles the Pythagorean Theorem can not be used to solve them.

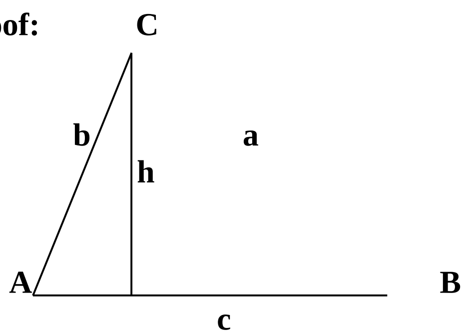
The Law of Sines:

If A, B and C are the measures of the angles of a triangle, and a,b and c are the lengths of the sides opposite these angles, then

$$\frac{a}{\sin a}=\frac{b}{\sin b}=\frac{c}{\sin c}$$

The ratio of the length of the side of any triangle to the sine of the angle opposite that side is the same for all three sides of the triangle.

Proof:



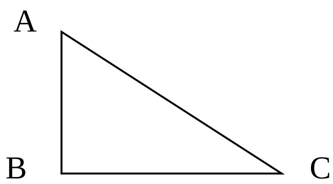
$$\left.\begin{aligned}\sin B=\frac{h}{a} &\quad\Rightarrow\quad h=a\sin B\\ \sin A=\frac{h}{b} &\quad\Rightarrow\quad h=b\sin A\end{aligned}\right\}\Rightarrow\quad a\sin B=b\sin A$$

$$\Downarrow$$

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$

The law of sines can be used as long as SAA or ASA are known to solve an oblique triangle.

1. Let $A = 64°$, $C = 82°$ and $c = 14$. Solve the right triangle shown below rounding to two decimals.



2. Solve the triangle ABC if $A = 40°$, $C = 22.5°$ and $b=12$.

Question: What happens when you know SSA?

This is an Ambiguous Case because there are three possible outcomes:

Since given two sides a,b and an angle A we might have:

No Triangle	**1 Triangle**	**2 Triangle**

3. Solve the triangle ABC if $A = 57°$, $a = 33$ and $b = 26$.

4. Solve the triangle ABC if $A = 35°$, $a = 12$ and $b = 16$.

5. Solve the triangle ABC if A = 50°, a = 10 and b = 20.

6. *Application*: Two fire lookout stations are 13meters apart, with B directly east of A. Both stations spot a fire. The bearing of the fire from A is N35°E and from B is N49°W. How far is the fire from station B?

Area of an Oblique Triangle.

7. Find the area of a triangle having two sides of 8m and 12m and an included angle of 135°.

Class 16: and Cosines

The law of cosines can be used as long as SAS or SSS are known to solve an oblique triangle.

The Law of Cosines:

If A, B and C are the measures of the angles of a triangle, and a,b and c are the lengths of the sides opposite these angles, then

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Solving an SAS triangle

1. Use the law of Cosines to find the side opposite the given angle
2. Use the Law of Sines to find the angle opposite the shorter of the two given sides (it will be acute)
3. Find the third angle by subtracting the measure of the given angle and the angle found in step 2 from 180°.

8. Solve the triangle with A=120°, b=7 and c=8.

Solving an SSS triangle
1. Use the law of Cosines to find the **angle** opposite the longest side.
2. Use the Law of Sines to find either of the two remaining acute angles.
3. Find the third angle by subtracting the angles from steps 1& 2 from 180°.

9. Solve the triangle ABC if a = 8, b = 10 and c = 5.

10. Application: Two airplanes leave the airport at the same time on different runways. One flies directly North at 400mph. The other on a bearing of N75°E at 350 mph. How far apart will the airplanes be after 2 hours?

Test 2: Summary/Questions

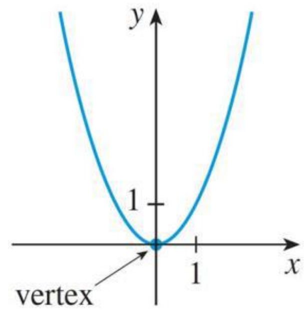
Class 18: Quadratic Functions

Objectives:

- Recognize Characteristics of Parabolas
- Graph Parabolas
- Determine a Quadratic Function's Max/Min Value
- Solve problems involving a quadratic function's max/min value.

Equations of Quadratic Functions

- **Basic quadratic function** is $f(x) = x^2$, whose graph is a parabola.
- **Vertex** is the lowest or highest point in the graph of a parabola



- **Standard Form of a Quadratic Equation**: The standard form of a quadratic function is

$$f(x) = a(x-h)^2 + k, \quad a \neq 0$$

  - The graph of a quadratic function is a parabola whose vertex is the point (h, k).
  - The parabola is symmetric with respect to the line $x = h$.
  - If $a > 0$, the parabola opens upward, If $a < 0$, the parabola opens downward.

Practice: Identify the vertex and axis of symmetry of each parabola below.

1. $f(x) = 3(x-2)^2 + 5$

2. $f(x) = 2(x+1)^2 - 4$

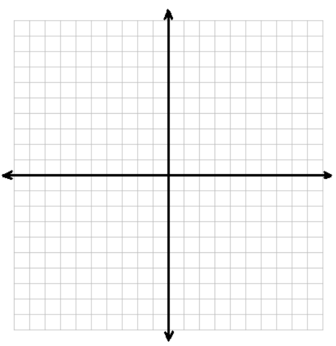
3. $f(x) = -7(x+2)^2$

4. $f(x) = 3x^2 - 1$

Graphing a Quadratic Function in Standard Form

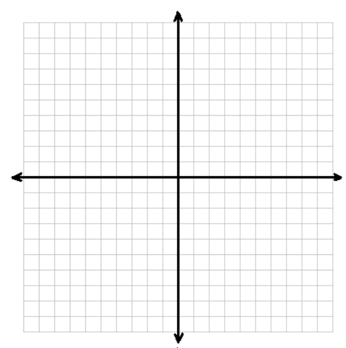
To graph $f(x) = a(x-h)^2 + k$

1. Determine whether the parabola opens upward ($a > 0$) or downward ($a < 0$).
2. Determine the vertex of the parabola (h, k).
3. Find the x-intercepts by solving $f(x) = 0$.
4. Find the y-intercepts by computing $f(0)$.
5. Plot all the above points and connect with a smooth curve.

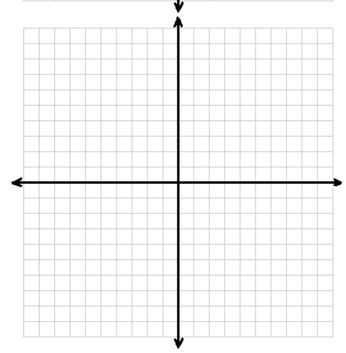


Practice: Graph the following quadratic functions

5. $f(x) = 2(x-3)^2 + 1$

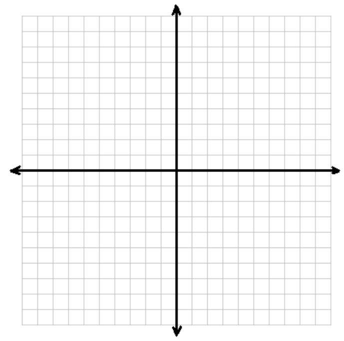


6. $f(x) = -3(x-4)^2 + 6$



7. $f(x) = 2(x+6)^2 - 2$

8. $f(x) = -(x-7)^2 + 3$



Graphing a Quadratic Function in the Form $f(x) = ax^2 + bx + c$

The vertex is $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$. The x-coordinate is $-\dfrac{b}{2a}$, and the y-coordinate of the vertex can be found by evaluating the function at $-\dfrac{b}{2a}$.

Practice: Find the vertex for each of the following quadratic functions.

9. $f(x) = x^2 + 3x - 10$

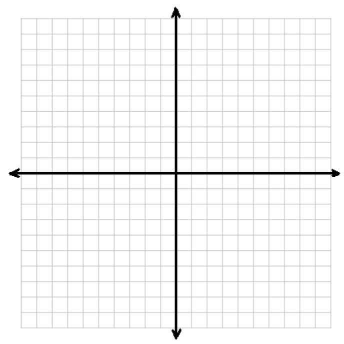
10. $f(x) = 2x - x^2 - 2$

To graph $f(x) = ax^2 + bx + c$

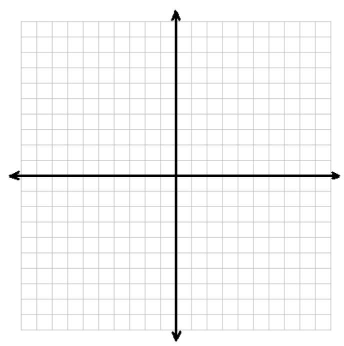
1. Determine whether the parabola opens upward ($a > 0$) or downward ($a < 0$).
2. Determine the vertex of the parabola
   $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$
3. Find the x-intercepts by solving $f(x) = 0$.
4. Find the y-intercepts by computing $f(0)$.
5. Plot all the above points and connect with a smooth curve.

Practice: Graph the following quadratic functions

11. $f(x) = 2x^2 + 4x - 3$



12. $f(x) = -4x^2 + 8x + 6$



Applications of Quadratic Functions

Maximum and Minimum Values of Quadratic Functions

Maximum and Minimum Values: Consider $f(x) = ax^2 + bx + c$

1. If $a > 0$, then f has a minimum that occurs at $-\frac{b}{2a}$
2. If ($a < 0$, then f has a maximum that occurs at $-\frac{b}{2a}$
3. The maximum/minimum value is $f\left(-\frac{b}{2a}\right)$

Practice: Find the min/max of each of the functions below.

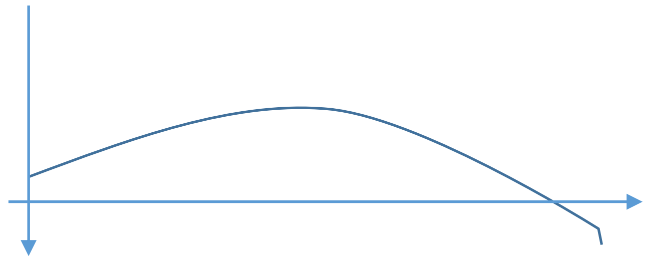
13. $f(x) = -2x^2 - 12x + 3$

14. $f(x) = 6x^2 - 6x$

15. $f(x) = 2x^2 - 8$

Applications of Quadratic Functions

1. When a shot is released at an angle of 65º, its height, g(x), in feet can be modeled by $g(x) = -0.04x^2 + 2.1x + 6.1$, where x is the shot's horizontal distance in feet, from its point of release. Use this model to answer the following:



a. What is the maximum height, to the nearest tenth of a foot, of the shot and how far from its point of release does this occur?

b. What is the shot's maximum horizontal distance, to the nearest foot, or the distance of the throw?

c. From what height was the shot released?

2. Among all pairs of numbers whose sum is 16, find a pair those product is as large as possible. What is the maximum product?

3. You have 200ft of fencing to enclose a rectangular plot that borders on a river. If you do not fence the side along the river, find the length and width of the plot that will maximize the area. What is the largest area that can be enclosed?

Class 19: Polynomial Functions & Division of Polynomials

Objectives:

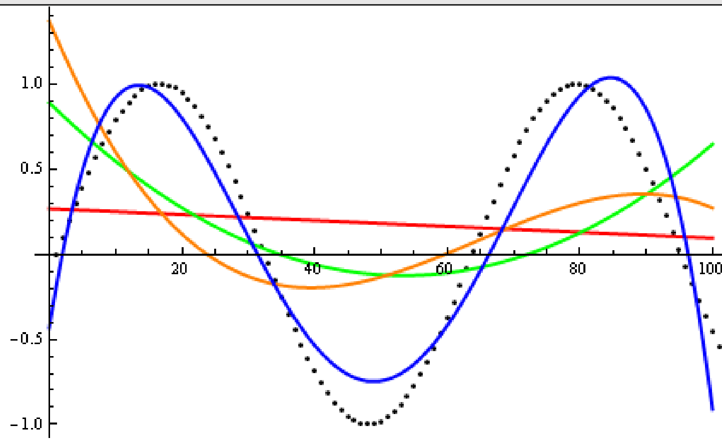
- Identify polynomial functions
- Recognize characteristics of graphs of Polynomial Functions
- Determine end behavior
- Identify zeroes and their multiplicities
- Use synthetic division to divide polynomials
- Use the Rational Zero Theorem to find possible rational zeros
- Find zeros of Polynomial Functions

Polynomial Functions

Let n be a nonnegative integer and let $a_0, a_1, a_2, \ldots, a_n$ be real numbers, with $a_n \neq 0$.

The function defined by $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots . a_2 x^2 + a_1 x + a_0$ is called a Polynomial Function of degree *n*. The number a_n the coefficient of the variable to the highest power, is called the leading coefficient.

The domain of polynomial functions is R = (-∞, ∞) and have graphs that are smooth and continuous.



1. Practice: Which of the following functions are polynomial functions?
   a. $f(x) = 3x^7 + 5x^3 + x^2 - 2$
   b. $f(x) = -2x^4 + 5x^{-2} + 8x$
   c. $f(x) = x^{2/3} + 7x^3 + \sqrt{x} - 2$

End Behavior of Polynomial Functions

- Polynomial functions of degree 2 or higher have graphs that are *smooth* and *continuous*.
- By **smooth**, we mean that the graphs contain only rounded curves with no sharp corners.
- By **continuous**, we mean that the graphs have no breaks and can be drawn without lifting your pencil from the rectangular coordinate system.

The behavior of the graph of a function to the far left or right is called the end behavior. The end behavior of the graph depends on **the degree** of the polynomial function $\boldsymbol{n}$ and the leading coefficient $\boldsymbol{a_n}$.

1. For n odd:

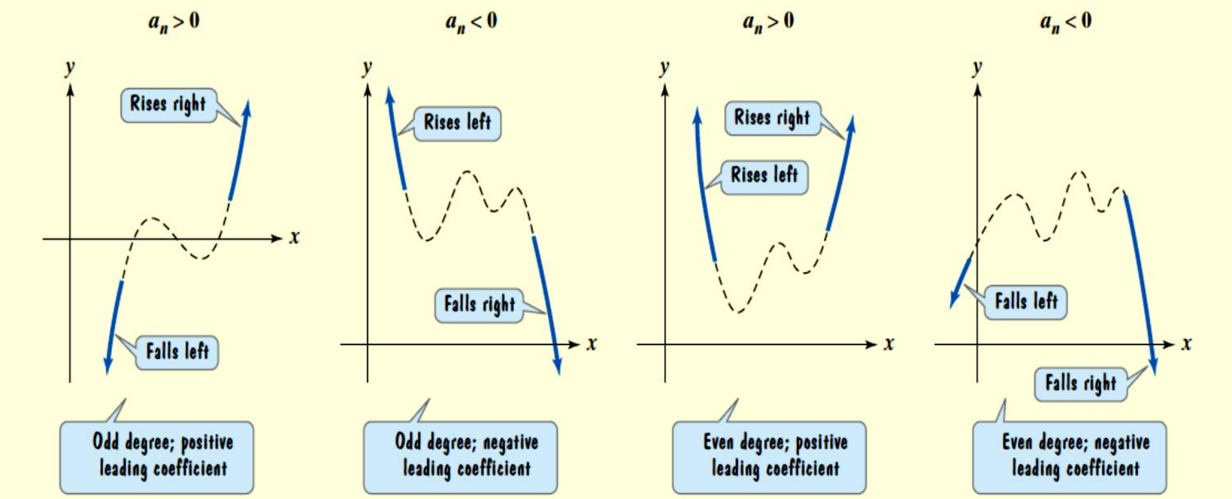
If the leading coefficient is positive, the graph falls to the left and rises to the right. (↙, ↗)

If the leading coefficient is negative, the graph rises to the left and falls to the right. (↖, ↘)

2. For n even:

If the leading coefficient is positive, the graph rises to the left and rises to the right. (↖, ↗)

If the leading coefficient is negative, the graph falls to the left and falls to the right. (↙, ↘)



Practice: Determine the end behavior of the following polynomial functions.

2. $f(x) = 5x^3 + 7x^2 - x + 5$

3. $f(x) = 2x^4 + x^3 - 4x^2 - 3x + 11$

4. $f(x) = -7x^5 + 4x^4 + 2x - 12$

5. $f(x) = -x^6 + 2x^5 + 7x^2 - 1$

Zeroes of Polynomial Functions

- If f is a polynomial function, then the values of x for which $f(x)$ is equal to 0 are called the zeros of f.
- These values of x are the roots, or solutions, of the polynomial equation $f(x) = 0$.
- Each real root of the polynomial equation appears as an x-intercept of the graph.
- If r is a zero of **even multiplicity**, then the graph **touches** the x-axis and turns around at r. If r is a zero of **odd multiplicity**, then the graph **crosses** the x-axis at r.

Practice: Find the zeros of the polynomial functions below

6. $f(x) = 2(x-3)^2(x+1)$

7. $f(x) = -3(x-2)^3(x+12)^2$

8. $f(x) = x^3 - 2x + x$

9. $f(x) = x^3 + 7x^2 - 4x - 28$

10. $f(x) = x^4 - 9x^2$

Synthetic Division

11. $\left(3x^2 - 5x + 5\right) \div \left(x - 5\right)$

12. $\left(x^3 + 6x^2 - 6x + 6\right) \div \left(x - 3\right)$

13. $\dfrac{2x^2 - 8x + 4x^3 + x^4}{4 + x}$

14. $\dfrac{\left(6x^5 - 2x^3 + 4x^2 - 5x + 5\right)}{x - 2}$

Use synthetic division to evaluate f(1) for $f(x) = 4x^3 - 12x^2 + 7x - 2$

Zeros of Polynomial Functions

The Rational Zero Theorem

If $f(x)=a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$ has integer coefficients and $\frac{p}{q}$ (in lowest terms) is a rational zero of *f*, then *p* is a factor of the constant term, a_0, and *q* is a factor of the leading coefficient a_n .

Practice: List all the possible rational zeroes of the following functions.

15. $f(x)=x^3+x^2-4x-4$

16. $f(x)=3x^4-11x^3-x^2+19x+6$

17. $f(x)=2x^4+3x^3-11x^2-9x+15$

Practice: Find all possible rational zeros and use long/synthetic division to test then and find the actual ones.

18. $f(x)=x^3+x^2-4x-4$

19. $f(x)=2x^3-3x^2-11x+6$

20. $f(x) = 2x^3 - 5x^2 + x + 2$

21. $f(x) = 3x^4 - 11x^3 - 3x^2 - 6x + 8$

Descartes Rule of Signs

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$ be a polynomials with real coefficients.

1. The number of positive real zeros of *f* is either
    a. The same as the number of sign changes of *f*(x), or
    b. Less than the number of sign changes of *f*(x) by a positive even integer. If f(x) has only one variation in sign, then *f* has exactly one positive real zero
2. The number of positive real zeros of *f* is either
    a. The same as the number of sign changes of *f*(-x), or
    b. Less than the number of sign changes of *f*(-x) by a positive even integer. If f(x) has only one variation in sign, then *f* has exactly one negative real zero

Practice: Find all the zeros of the polynomial functions

22. $f(x) = x^3 - 4x^2 - 7x + 10$

23. $f(x) = x^3 - 6x^2 - 9x + 14$

24. $f(x) = x^3 - 3x^2 - 33x + 35$

Class 21: Rational Functions

Objectives:

- Find the Domain of Rational Functions
- Identify Vertical Asymptotes
- Identify Horizontal Asymptotes
- Applications of Rational Functions

➢ Rational Functions

Rational Functions are quotient of polynomials i.e., functions that can be expressed as a fraction $f(x)=\dfrac{p(x)}{q(x)}$ where p and q are polynomials and $q(x)\neq 0$

Example: $f(x)=\dfrac{x^2+2x+9}{(x-3)(x^2-16)}$

- **Domain:**

Domain of a R. F. are all real numbers except those that make the denominator equal to 0.

Example: *Find the domain of* $f(x)=\dfrac{x^2+2x+9}{(x-3)(x^2-16)}$

To find the domain we set the denominator equal to 0 and solve for x.

$(x-3)(x^2-16)=0\Rightarrow$

$x-3=0 \quad or \quad x^2-16=0\Rightarrow$

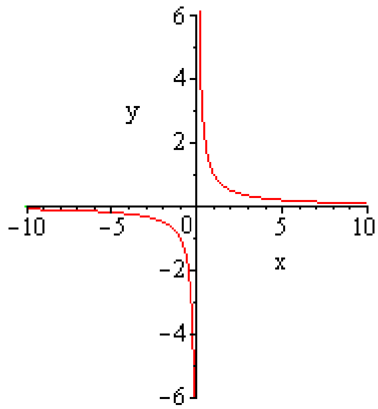
$x=3 \qquad (x+4)(x-4)=0 \Rightarrow$

$x=\pm 4$

Hence the domain is $\{x\in\mathbb{R}\,|\,x\neq 3,\pm 4\}$ or $(-\infty,-4)\cup(-4,3)\cup(3,4)\cup(4,\infty)$

Always factor the denominator first, in order to find the domain.

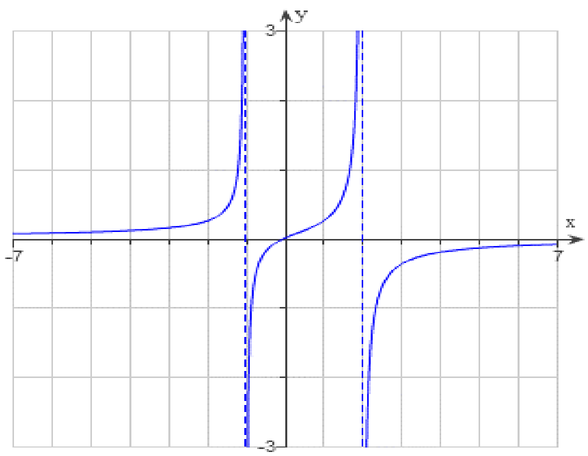
The most basic rational function is the inverse function $f(x) = \dfrac{1}{x}$



$Range:$

$X-\text{int}:$

Practice: Discribe the behavior of the function at -1 and at 2.



$As \quad x \to 2^{+} \qquad f(x) \to$

$As \quad x \to 2^{-} \qquad f(x) \to$

$As \quad x \to -1^{+} \qquad f(x) \to$

$As \quad x \to -1^{-} \qquad f(x) \to$

Asymptotes

- **Vertical**: to find the VA set the denominator equal to 0.
- **Horizontal**: to find the HA we compare the degrees of the Numerator and Denominator.

- If degNum < degDen then the HA is y=0
- If degNum > degDen then there is **NO** HA
- If degNum = degDen then the HA is $y = \dfrac{a_n}{b_n}$ (the quotient of leading coefficients)

Practice: Find all Asymptotes for each of the following functions

1. $f(x)=\dfrac{x+2}{x^2+x-6}$

2. $f(x)=\dfrac{x^3-27}{x-3}$

3. $f(x)=\dfrac{x^3+1}{x^2+2x}$

4. $f(x)=\dfrac{2x^3+1}{3x^3+2x}$

5. $f(x)=\dfrac{3x^2+1}{x^2-16}$

Characteristics and Graphs of Rational Functions

What do we need to know about R.F. ?

- Domain and Range
- Asymptotes (VA, HA,SA)
- x and y-intercepts
- Local max and min
- Directional Limits

Practice: *Give the characteristics and sketch a graph for each of the following functions.*

4. $f(x)=\dfrac{2x+1}{x+1}$

- Domain:
- Range:
- x-int:
- y-int:
- HA:
- VA:
- SA:

D. Limits:

5. $f(x)=\dfrac{x}{x^2-1}$

- Domain:
- Range:
- x-int:
- y-int:
- HA:
- VA:
- SA:

D. Limits:

6. $f(x)=\dfrac{x^3+3x^2-4x-12}{2x^2-8}$

- Domain:
- Range:
- x-int:
- y-int:
- HA:
- VA:
- SA:

D. Limits:

7. $f(x)=\dfrac{x^3-27}{x^2+5x-6}$

- Domain:
- Range:
- x-int:
- y-int:
- HA:
- VA:
- SA:

D. Limits:

Variation

- y varies Directly as x if $y = kx$
- y varies Inversely as x if $y = \frac{k}{x}$
- y varies Jointly to x and z $y = kxz$

1. The amount of gas that a helicopter uses is directly proportional to the number of hours spent flying. The helicopter flies for 3 hours and uses 24 gallons of fuel. Find the number of gallons of fuel that the helicopter uses to fly for 6 hours.

2. The weight of a body above the Earth's surface varies inversely with the square of the distance from the center of the Earth. If a certain body weighs 55 pounds when it is 3960 miles from the center of the Earth, then how much will it weigh when it is 3965 miles from the center of the Earth?

3. While traveling at a constant speed in a car, the centrifugal acceleration passengers feel while the car is turning is inversely proportional to the radius of the turn. If the passengers feel an acceleration of 4 ft per second per second when the radius of the turn is 90 ft, find the acceleration the passengers feel when the radius of the turn is 180 ft.

4. The volume of a gas in a container varies directly as its temperature and inversely as its pressure. At a temperature of 75 degrees Kelvin and a pressure of 15 kilograms per square meter, the gas occupies a volume of 25 cubic meters. Find the volume at a temperature of 125 degrees Kelvin and a pressure of 35 kilograms per square meter

5. The amount of paint needed to cover the walls of a room varies jointly as the perimeter of the room and the height of the wall. If a room with a perimeter of 45 feet and 8 foot walls requires 3.6 quarts of paint, find the amount of paint needed to cover the walls of a room with a perimeter of 75 feet and 10-foot walls

Class 22: Exponential and Logarithmic Functions

Definition: The exponential function f with base b is defined by $f(x) = b^x$ where b is a positive constant other than 1 (b>0 and b≠1) and x is a real number.

These functions are used to model rapidly increasing or decreasing situations
Population Growth, Growth of Epidemics, Radioactive Decay, Cooling or Heating

Basic Laws of Exponents

$a^1 = a$ $\quad a^0 = 1$ $\quad a^x a^y = a^{x+y}$ $\quad \dfrac{a^x}{a^y} = a^{x-y}$ $\quad (a^x)^y = a^{xy}$

$(ab)^x = a^x b^x$ $\quad \left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$ $\quad a^{x/y} = \sqrt[y]{a^x}$

Simplify the following:

1. $3^2 3^5 =$
2. $2^3 + 2^2 =$
3. $\dfrac{4^{13}}{4^{15}} =$
4. $13^0 =$
5. $7^1 =$
6. $\dfrac{2^{27}}{2^{24}} =$
7. $(2^2)^3 =$
8. $(5)^{-3} =$
9. $(16)^{3/2} =$
10. $(7 \cdot 3^3)^2 =$
11. $(2x^3 y^4)^5 =$
12. $\left(\dfrac{1}{3}\right)^{-3} =$
13. $\sqrt[5]{(32)^3} =$
14. $(27)^{1/3} =$

Evaluating

1. In college, we study large volumes of information that, unfortunately we do not often retain for very long. The function $f(x) = 80e^{-0.5x} + 20$ describes the percentage of information that a person can be expected to remember x weeks after learning it.

a. Let x = 0 and give the value of f(0)

b. Let x = 52 and determine the value of f(52) accurate to the nearest ten thousandth

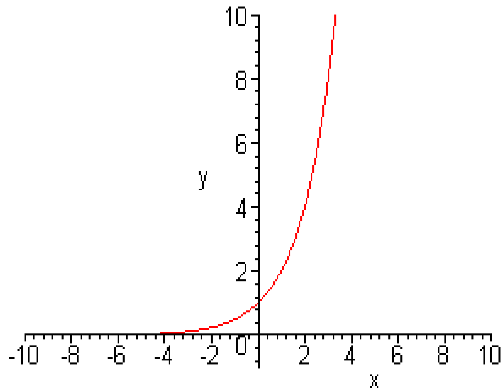
2. The function $f(x) = 13.49(0.967)^{x} - 1$ describes the number of O-rings expected to fail, when the temperature is x^{o} F. On the morning the Challenger was launched, the temperature was 31^{o}F, colder than any previous experience.

a. Find the number of O-rings expected to fail at this temperature. Round to the nearest whole number

b. Find the number of O-rings expected to fail at a temperature of 60^{o}F. Round to the nearest whole number

Exponential Functions

- **Exponential Functions:** What does an exponential function look like → $f(x) = a^x$



- ***y-intercept***: (0,1) because $f(0) = a^0 = 1$

Domain: x-values→ (- ∞, ∞)

Range: → (0, ∞)

- ***Asymptotes***?
  The graph approaches but never touches the x-axis so y = 0 is a horizontal asymptote.

- ***Inverse***
  $f(x) = a^x$ is one-to-one and has an inverse function $f^{-1}(x) = \log_a x$

<u>Transformations Involving Exponential Functions</u>

Transformation	Equation	Description
Vertical translation	$f(x) = b^x + c$ $f(x) = b^x - c$	**Shifts the graph upward c units** **Shifts the graph downward c units**
Horizontal translation	$f(x) = b^{x+c}$ $f(x) = b^{x-c}$	**Shifts the graph to the left c units** **Shifts the graph to the right c units**
Reflection	$f(x) = -b^x$ $f(x) = b^{-x}$	**Reflects the graph about the x-axis** **Reflects the graph about the y-axis**
Vertical Stretching/ Shrinking	$f(x) = cb^x$	**Vertically stretch if c >1 and Vertically Shrinks if 0 <c<1**
Horizontal Stretching/Shrinking	$f(x) = b^{cx}$	**Horizontally Shrinks if c >1 and Horizontally Stretches if 0 < c < 1**

<u>Sketch the graphs of the following functions using transformations?</u>

17. $f(x) = 2^{-x}$ 18. $f(x) = 2^{x-3}$ 19. $f(x) = 2^x - 3$

Compounding

- **Compound Interest** is interest compounded on your original investment as well as the on any accumulated interest. The accumulated amount A is given by the following formula

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Where, **P** is the **Principal,** ***r*** is the annual percentage rate, ***n*** is the number of compoundings and ***t*** –years.

- **The Natural Base *e*** : If the number of compundings goes to infinity then the quantity

$\left(1+\frac{r}{n}\right)^{nt} \to e$ ***e*** is an irrational number. It's value is approximately 2.718281827.

- **Continuous Compounding:** Continuous compounding is given by the formula $A = Pe^{rt}$

20. Laura borrows $2500 at a rate of 10.5%. Find how much Laura owes at the end of 4 years if:

    a. The interest is compounded yearly

    b. The interest is compounded quarterly

    c. The interest is compounded monthly

    d. The interest is compounded continuously

    e. Which option would yield the most interest, 10.5% compounded monthly for 4 years or 9% compounded continuously?

Logarithmic Functions

- **Definition:** For x > 0 and b >0, b≠1, $y = \log_b x$ is equivalent to $b^y = x$
  $f(x) = \log_b x$ is the logarithmic function with base b.

A logarithm y is an exponent: $y = \log_b x \quad \Leftrightarrow \quad b^y = x$

- **Basic Laws of Logarithms**

$$\log_b 1 = 0 \qquad \log_b b^x = x \qquad \log_b b = 1 \qquad b^{\log_b x} = x$$

Write the following in its equivalent exponential form:

2. $4 = \log_2 16$
2. $6 = \log_2 64$
3. $\log_6 216 = y$
4. $\log_5 125 = y$

Write the following in its equivalent logarithmic form:

5. $\sqrt[3]{8} = 2$
6. $13^2 = x$
7. $5^{-3} = \dfrac{1}{125}$
8. $7^y = 200$

Evaluate the following without a calculator:

9. $\log_7 49 =$

10. $\log_6 \frac{1}{6} =$

11. $\log_7 \sqrt{7} =$

12. $\log_2 \frac{1}{\sqrt{2}} =$

13. $\log_4 1 =$

14. $\log_5 5^7 =$

15. $8^{\log_8 19} =$

16. $7^{\log_7 23} =$

Natural Logarithms lnx

$f(x) = \ln x$ is the logarithmic function with base e is called the Natural Logarithmic function.

$$\boxed{\log_e x = \ln x}$$

$f(x) = e^x$ is the inverse function of $f(x) = \ln x$

Properties of ln(x).

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^x = x$
4. $e^{\ln x} = x$

Simplify the following

1. $\ln e^6$

2. $\log 10^7$

3. $\ln \frac{1}{e^6}$

4. $10^{\log 53}$

5. $e^{\ln 125}$

6. $e^{\ln 7x^2}$

- **Logarithmic Functions:** What does a logarithmic function look like → $f(x) = \log_a x$

  - ***y-intercept***: (1,0) because $f(1) = \log_a 1 = 0$
  - ***Domain***: x-values→ (0, ∞)
  - ***Range***: → (-∞, ∞)
  - ***Asymptotes***
    The graph approaches but never touches the y-axis so x = 0 is a vertical asymptote.
  - ***Inverse***?
    $f(x) = \log_a x$ is one-to-one and has an inverse function $f^{-1}(x) = a^x$

Transformations Involving Logarithmic Functions

Transformation	**Equation**	**Description**
Vertical translation	$f(x) = \log_b x + c$ $f(x) = \log_b x - c$	**Shifts the graph upward c units** **Shifts the graph downward c units**
Horizontal translation	$f(x) = \log_b (x + c)$ $f(x) = \log_b (x - c)$	**Shifts the graph to the left c units** **Shifts the graph to the right c units**
Reflection	$f(x) = -\log_b x$ $f(x) = \log_b (-x)$	**Reflects the graph about the x-axis** **Reflects the graph about the y-axis**
Vertical Stretching/ Shrinking	$f(x) = c \log_b x$	**Vertically stretch if c >1 and Vertically Shrinks if 0 <c<1**
Horizontal Stretching/Shrinking	$f(x) = \log_b (cx)$	**Horizontally Shrinks if c >1 and Horizontally Stretches if 0 < c < 1**

What do the graphs of the following functions look like?

17. $f(x) = \ln(x-1)$

18. $f(x) = \ln(x) - 1$

19. $f(x) = \ln(x+3) + 2$

20. Find the domain of the following functions:

a. $f(x) = \ln(x-2)$

b. $f(x) = \log(3x+6)$

21. The percentage of adult height attained by a girl who is x years old can be modeled by $f(x) = 62 + 35\log(x-4)$ where x represents the girl's age and f(x) represents the percentage of her adult height.

a. Approximately what percentage of her adult height has a girl attained at age 13?

b. Approximately what percentage of her adult height has a girl attained at age 16?

21. Students in a psychology class took a final examination. As part of an experiment to see how much of the course content they remembered over time, they took equivalent forms of the exam in monthly intervals thereafter. The average score for the group, *f(t)*, after *t* months was modeled by the function $f(t) = 88 - 15\ln(t+1)$, $0 \le t \le 12$.

a. What was the average score on the original exam?

b. What was the average score after 2 months, 8 months, one year?

Class 23: Exponential and Logarithmic Equations and Logistic Growth

- **Exponential Equation** is an equation that contains an exponential expression.

Example: $3^{x-5} = 81$ or $26e^{0.12x} = 320$

Practice: Solve the following equations

1. $4^{2x-1} = 64$

2. $3^{x-1} = \dfrac{1}{27}$

3. $e^{x+4} = \dfrac{1}{e^{2x}}$

Steps to solve exponential equations

- Isolate the exponential expression in one side and everything else on the other
- Take Logarithm on both sides of the equation
- Simplify using one of the following properties $\log_b b^x = x$ or $\ln e^x = x$
- Solve for the variable

Example:
Solve for x
$5^x = 134$

$$5^x = 134 \Rightarrow \qquad \textit{or} \qquad 5^x = 134 \Rightarrow$$

$$\log_5 5^x = \log_5 134 \Rightarrow \qquad \ln 5^x = \ln 134 \Rightarrow$$

$$x = \log_5 134 \Rightarrow \qquad x \ln 5 = \ln 134 \Rightarrow$$

$$\boxed{x = \frac{\log 134}{\log 5}} \approx 3.0432 \qquad \boxed{x = \frac{\ln 134}{\ln 5}}$$

Solve the following equations

4. $3e^{5x} = 1977$

5. $e^{4x-5} - 7 = 11{,}243$

6. $5e^{0.002x} - 9 = 12$

7. $e^{x-2} - 12^{2x+3} = 0$

8. The formula $A = 15.9e^{0.0235t}$ models the population of Florida, A, in millions, t years after 2000.
   a. What was the population of Florida in 2000?

   b. When will the population reach 19.2 Million?

9. The function $f(x) = 6.4e^{0.0123x}$ describes world population, f(x) in billions, x years after 2004 subject to a growth rate of 1.23% annually.
   a. Use the function to predict world population in 2050.

   b. When will the population reach 10 million?

Logistic Growth

Nothing in earth grows exponentially indefinitely. Every growth is limited because of factors such as death rate, food supplies, space etc. Logistic Growth is used for such situations.

- **Logistic Growth Model** $A = \dfrac{c}{1+ae^{-bt}}$ where a, b, c are constants and c > 0, b > 0

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7. The logistic growth function $f(t) = \dfrac{100,000}{1+500e^{-t}}$ describes the number of people, f(t), who have become ill with influenza t weeks after its initial outbreak in a particular community.

a. How many people became ill with the flu when the epidemic began?

b. How many people were ill by the end of the fourth week?

c. What is the limiting size of the population that becomes ill?

Properties of Logarithms

- **Product Rule**: $\boxed{\log_b(MN) = \log_b(M) + \log_b(N) \qquad b, M, N \in \mathbb{R}^+, \quad b \neq 1}$

  Example: $\log\left[(x+2)(x+3)\right] = \log(x+2) + \log(x+3)$

  Practice: Simplify

  1. $\ln(2x) =$

  2. $\ln(x+3) + \ln(x-1)$

- **Quotient Rule**: $\boxed{\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N) \qquad b, M, N \in \mathbb{R}^+, \quad b \neq 1}$

  Example: $\log\left[\frac{(2x+3)}{(x-5)}\right] = \log(2x+3) - \log(x-5)$

  Practice: Simplify

  3. $\ln(x^2) - \ln(x) =$

  4. $\ln(x^2-4) - \ln(x-2) =$

  5. $\log(x^2-3x-4) - \log(x+1) =$

- **Power Rule**: $\boxed{n\log_b(m) = \log_b(m^n) \qquad b, M, N \in \mathbb{R}^+, \quad b \neq 1}$

  Example: $7\log(x-3) = \log(x-3)^7$

  Practice: Simplify

  6. $\ln(x^3) =$

  7. $\frac{1}{2}\ln x^4 =$

  8. $3\log(x^2-3x-4) =$

- **Rules for solving Logarithmic equations**

- **Move all log terms to the one side and everything else to the other side**
- **Use the log. properties to compress the log side to only one term of log(exp.)**
- **Compose each side in the inverse function for the logarithmic function**
- **Solve the equality for the variable using algebra**
- **Check the answers.**

Example: Solve $\ln(x+3) = 7 + \ln(x)$

$$\ln(x+3) = \ln(x) + 7 \Rightarrow \ln(x+3) - \ln(x) = 7 \quad [\textit{get all } \log \textit{ terms in one side}]$$

$$\ln\left(\frac{x+3}{x}\right) = 7 \Rightarrow \quad [\textit{compress all terms in one } \log \textit{ expression}]$$

$$e^{\ln\left(\frac{x+3}{x}\right)} = e^7 \Rightarrow \quad [\textit{compose both sides with the inverse function}]$$

$$\left.\begin{array}{l} \dfrac{x+3}{x} = e^7 \Rightarrow x+3 = xe^7 \\ 3 = xe^7 - x \Rightarrow \\ 3 = x(e^7 - 1) \Rightarrow \end{array}\right\} [\textit{solve the equation for } x]$$

$$x = \frac{3}{e^7 - 1} \quad or \quad x = 0.0027$$

Practice: Solve

9. $\log_5 x = 3$

10. $\log_7 (x+2) = -2$

11. $6\ln(2x) = 30$

12. $7+3\ln x=6$

13. $\ln\sqrt{x+3}=1$

14. $\log_4(x+2)-\log_4(x-1)=1$

Test 3: Summary/Questions